GALERKIN-WAVELET METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. Anti-derivatives of wavelets are used for the numerical solution of differential equations. Optimal error estimates are obtained in the applications to two-point boundary value problems of second order. The orthogonal property of the wavelets is used to construct efficient iterative methods for the solution of the resultant linear algebraic systems. Numerical examples are given.

Key words. wavelets, finite element method, two-point boundary value problem, conjugate gradient method, preconditioner, piecewise linear hierarchical basis

Subject classifications: AMS(MOS): 65N30, 65N13, 65F10; CR: G1.8.

1. Introduction. In this paper, we shall study wavelet functions applying to the numerical solution of differential equations. Wavelets in our considerations are those discovered by Daubechies in 1988 [7], which have compact supports and form an orthonormal basis of $L^2(\mathbb{R})$.

The idea of wavelets may be traced back to Calderon [2] and Coifman and Weiss [4]. Modern ideas grow out of applications in signal processing, cf. Liénard [19], Rodet [24] and Goupillaud, Grossmann and Morlet [12]. Earlier mathematical analysis of wavelets can be found in Stromberg [26], Grossmann and Morlet [14] and Meyer [22]. In recent years, wavelets have been studied by Meyer and his group using the multiresolution framework, cf. Mallat [20] and Meyer [23]. The multiresolution approach has lead to the discovery of a family of wavelets with compact supports by Daubechies in 1988 [7]. For the application of wavelets to practical problems, we refer to [5, 6, 10, 13, 17, 18, 21].

Applying wavelets to discretize differential equations appears to be a very attractive idea. In finite element type methods, piecewise polynomial trial functions may be replaced by wavelets. Such an idea was explored by Glowinski, Lawton, Ravachol and Tenenbaum [10] (see also the unpublished technical reports cited therein). Many interesting numerical examples in their paper suggest that wavelets have great potential in the application to the numerical solution of differetial equations.

At this stage of research, a thourough study of one dimensional problems is still necessary and theoretically important. The simplest case would be the two-point boundary value problem for the second order elliptic equation, but how the wavelets can be appropriatly used in this case is still not clear. If the wavelets are used directly as trial functions, as is done in [10], several difficulties would be encountered. First, due to the lack of regularity, "lower order" wavelets can not be employed. Secondly, Dirichlet boundary value problems can not be applied directly because the boundary conditions are hard to impose on subspaces. Thirdly, the orthogonality, one of the main features of wavelets, does not play any significant roles.

In this paper, we shall take a different approach. Instead of using wavelets directly, we take their anti-derivatives as trial functions. In this way, singularity in the wavelets is smoothed, the boundary condition can be treated easily and the orthogonality is used to construct efficient algorithms to solve the underlying algebraic system. Moreover, we can develop a rather complete theory in most important aspects of our algorithm.

Our approach was motivated by the observation that the differentiation of the so-called hierarchical basis functions (cf. [27, 28]) in one dimensional piecewise linear finite elements are exactly

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the wavelets of order 1, namely the Haar basis [15]. Taking the anti-derivatives of higher order wavelets then leads to higher order trial functions.

This work is our first attempt to use wavelets for numerical solution of differential equations. For one-dimensional problems studied in this paper, the wavelets approach appears to be attractive and our theory is rather complete. Applications to higher dimensional problems (on the domains other than rectangles) remain to be explored and the prospect of the method is yet to be seen.

We shall use the standard notation $L^2(\mathbb{R})$ to denote the space of square integrable functions Two functions $u, v \in L^2(\mathbb{R})$ are orthogonal in $L^2(\mathbb{R})$ if $(u, v) = 0$. Given $\Omega = (a, b)$ $(-\infty \le a < b \le \infty)$, $H^s(\Omega)$ denotes the standard Sobolev space with the norm $\|\cdot\|_{s,\Omega}$ and semi-norm $|\cdot|_{s,\Omega}$ given by

$$\|v\|_{s,\Omega}^2 = \sum_{i=0}^{s} \int_a^b |v^{(i)}(x)|^2\,dx \quad \text{and} \quad |v|_{s,\Omega}^2 = \int_a^b |v^{(s)}(x)|^2\,dx.$$

For $-\infty < a < b < \infty$, we define

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v(a) = v(b) = 0\} \quad \text{and} \quad H_*^1(\Omega) = \{v \in H^1(\Omega) \mid v(a) = 0\}.$$

It is well-known that the semi-norm $|\cdot|_{1,\Omega}$ is a norm (equivalent to $\|\cdot\|_{1,\Omega}$) in these two spaces. When $\Omega = \mathbb{R}$, we denote $\|v\|_s = \|v\|_{s,\Omega}$ and $|v|_s = |v|_{s,\Omega}$.

Throughout this paper, we shall use the letter C to denote a generic positive constant which may stand for different values at its different appearances. When it is not important to keep track of these constants, we shall conceal the letter C into the notation $\lesssim$, $\eqsim$, or $\gtrsim$. Here

$$x \lesssim y \quad \text{means} \quad x \le Cy, \tag{1.1}$$

$x \gtrsim y$ means $-x \lesssim -y$, and $x \eqsim y$ means $x \lesssim y$ and $x \gtrsim y$.

The rest of the paper is organized as follows. Section 2 gives a brief introduction to wavelet functions. In Section 3, a simple technique by means of anti-derivatives is presented to construct a frame in the space H^1 from a frame in the space L^2; the optimal approximation properties of the frames in H_0^1 (or H_*^1, H^1) based on wavelets are established. Applications to the numerical solution of a second order two-point boundary value problem is given in Section 4. This section concerns the optimal error estimates for the approximate solution, the selection of basis functions, effective iterative techniques for solving the underlying linear algebraic system and finally some comparisons with the standard hierarchical basis method. Finally, in Section 5, a number of numerical examples are given to support our theory.

2. Basic properties of wavelets. In this section, we shall give a brief introduction to the constructions and basic properties of wavelets. More detailed discussions can be found in [20, 23, 25].

Given a positive integer p, consider a set of constants c_k that satisfy the following four properties:

$$c_k = 0 \text{ for } k \notin \{0, 1, \ldots, 2p-1\}, \tag{2.1}$$

$$\sum_k c_k = 2, \tag{2.2}$$

$$\sum_k (-1)^k k^m c_k = 0, \quad \text{for } 0 \le m \le p-1, \tag{2.3}$$

and

$$\sum_k c_k c_{k-2m} = 2\delta_{0m}, \quad \text{for } 1-p \le m \le p-1. \tag{2.4}$$

Solutions of c_k for $1 \le p \le 10$ that satisfy (2.1)–(2.4) can be found in [7] (the coefficients $h(n)$ are listed on p. 980, while $c_n = \sqrt{2}h(n)$). In particular, for $p = 1$,

$$c_0 = 1, \quad c_1 = 1,$$

and, for $p = 2$,

$$c_0 = \frac{1+\sqrt{3}}{4}, \quad c_1 = \frac{3+\sqrt{3}}{4}, \quad c_2 = \frac{3-\sqrt{3}}{4}, \quad c_3 = \frac{1-\sqrt{3}}{4}.$$

For each $p \ge 1$ and the corresponding coefficients c_k determined above, we consider the so called *dilation equation*

$$\phi(x) = \sum_k c_k \phi(2x-k). \tag{2.5}$$

It is shown in [7] that (2.5) has a unique solution such that supp$\phi = [0, R]$, with $R = 2p-1 > 0$. There are several ways to construct the solution $\phi(x)$ (cf. [23, 25]). For example, we can take ϕ to be the limit of the iteration

$$\phi^{n+1}(x) = \sum_k c_k \phi^n(2x-k), \quad n = 0, 1, \ldots, \tag{2.6}$$

where ϕ^0 is the unit indicator function $\chi_{[0,1]}$. For $p = 1$, ϕ^0 is invariant in (2.6) and hence $\phi(x) = \chi_{[0,1]}$. For $p = 2$, $\phi(x) \in C^r$ where r is about 0.55 (Figure 1). In general, $r = r(p)$ is an increasing function [8] and $r(p) \to \infty$ linearly as $p \to \infty$ [7].

Given $\phi(x)$ (the solution of (2.5)), we define

$$\tilde{\psi}(x) = \sum_k (-1)^k c_{1-k}\phi(2x-k) \quad \text{and} \quad \psi(x) = \tilde{\psi}(x-p+1). \tag{2.7}$$

Note that supp$\tilde{\psi} = [1-p, p]$ and supp$\psi = [0, R]$. Usually the function $\tilde{\psi}$ is used to define wavelets, but for our purpose ψ will be used instead. For $p = 1$, ψ is the Haar function [15]. Pictures of ψ for the cases $p = 1$ and 2 are shown in Figures 2 and 3, respectively.

We are now in a position to present the definition of wavelets.

DEFINITION 2.1. *Assume $p \ge 1$ is a given integer. Let c_k be determined by* (2.1)–(2.4), *ϕ satisfy* (2.5) *and ψ be given in* (2.7). *Define*

$$\psi_{jk}(x) = \begin{cases} \phi(x-k), & \text{for } j = -1, \\ \sqrt{2^j}\,\psi(2^j x - k), & \text{for } j \ge 0. \end{cases} \tag{2.8}$$

The functions ψ_{jk} with $j \ge -1$ and $(j,k) \in \mathbb{Z} \times \mathbb{Z}$ are called wavelets of order p.

PROPOSITION 2.1. (1) *The set of wavelets forms an orthonormal basis of $L^2(\mathbb{R})$. That is,*

$$(\psi_{jk}(x),\, \psi_{\ell m}(x)) = \delta_{j\ell}\delta_{km} \tag{2.9}$$

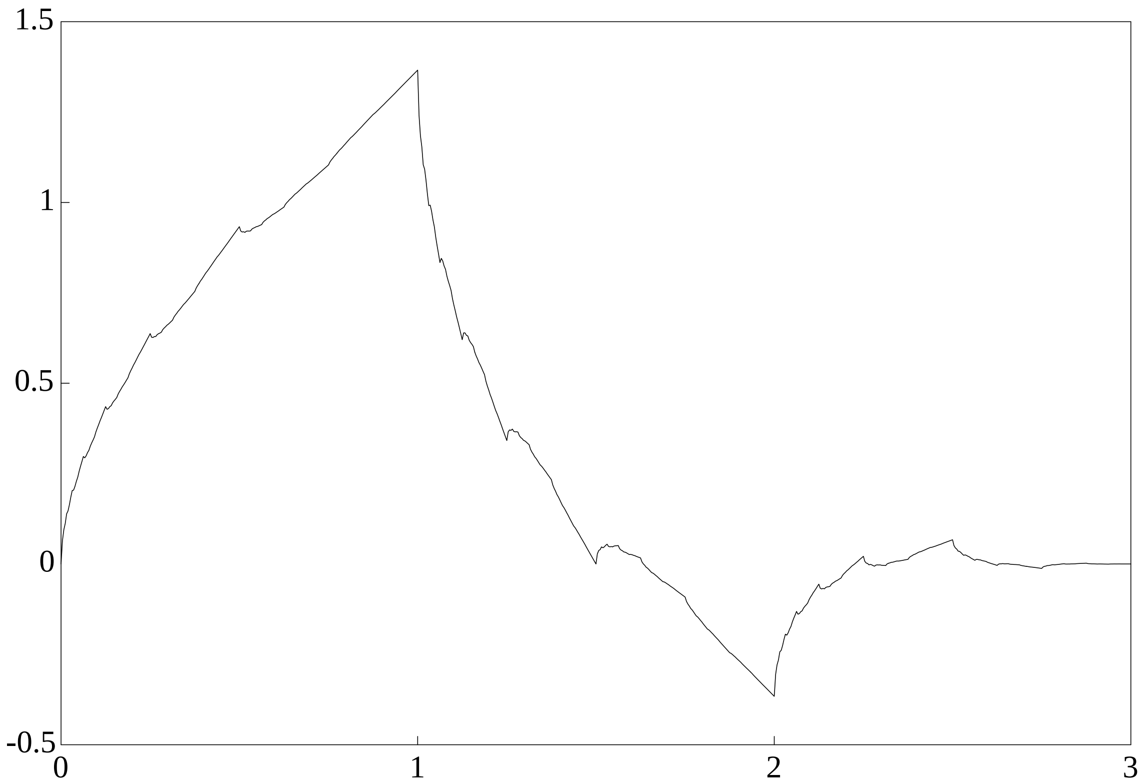

Figure 1. $\phi(x)$ for $p=2$.

and

$$\lim_{J\to\infty} \|u - \sum_{j=-1}^{J} \sum_k \alpha_{jk}\psi_{jk}\| = 0, \quad \textit{for all } u \in L^2(\mathbb{R}),$$

where

$$\alpha_{jk} = (u, \psi_{jk}). \tag{2.10}$$

(2) *For $j \ge 0$, the first p moments of ψ_{jk} equal to zero:*

$$\int_{-\infty}^{\infty} x^m \psi_{jk}(x)\,dx = 0, \quad 0 \le m \le p-1. \tag{2.11}$$

We shall now give a brief illustration for the assertions in the above proposition. For detailed proofs, we refer to [7, 23].

By (2.2) and (2.5), we have $\int_{-\infty}^{\infty} \phi\,dx = 1$. By (2.4), (2.6) and induction we get $(\phi(x), \phi(x-k)) = \delta_{0k}$, which together with (2.7) show that $(\phi(x), \tilde\psi(x-k)) = 0$. Then an application of (2.4) and (2.7) yields $(\tilde\psi(x), \tilde\psi(x-k)) = \delta_{0k}$. Thus (2.9) is proved for $\{j,\ell\} \subset \{-1,0\}$. The general case with $j=\ell$ follows from a proper change of variables of x. Finally, by (2.5) and induction we can prove generally with $j<\ell$.

To see (2.11), let $\hat{\tilde\psi}(\omega)$ be the Fourier transform of $\tilde\psi(x)$, $P(\omega) = \frac{1}{2}\sum c_k e^{ik\omega}$ and $Q(\omega) = \frac{1}{2}\sum(-1)^k\, c_{1-k}e^{ik\omega}$. Since $\hat\phi(0)=1$, we have formally

$$\hat{\tilde\psi}(\omega) = Q(\frac{\omega}{2}) \cdot \prod_{n=2}^{\infty} P(\frac{\omega}{2^n}).$$

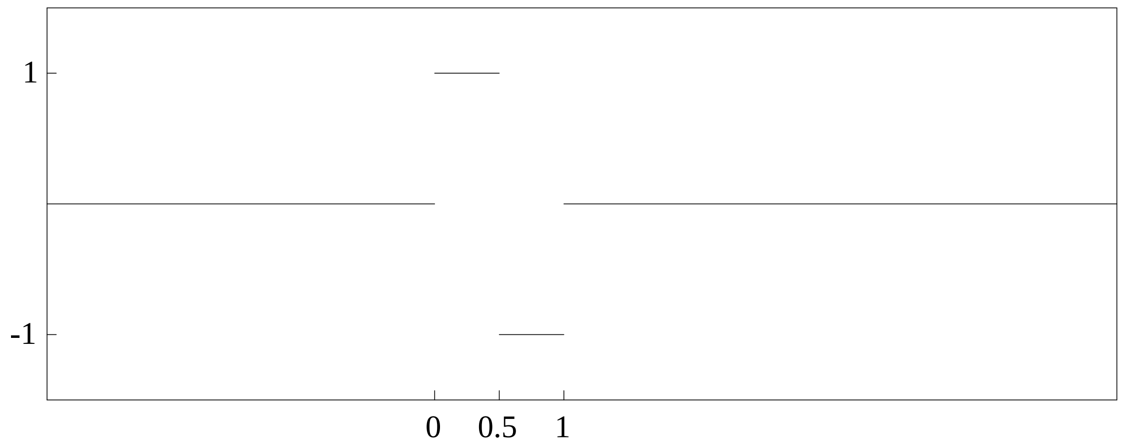

Figure 2. $\psi(x)$ for $p = 1$.

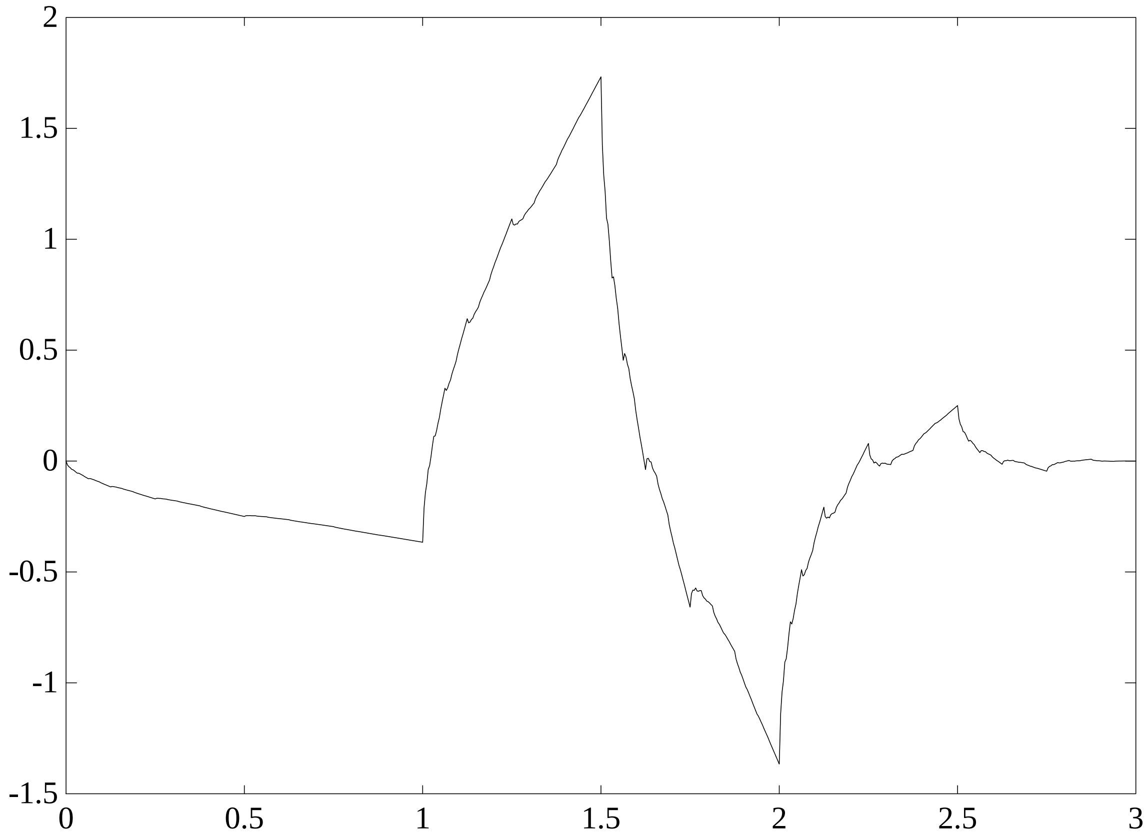

Figure 3. $\psi(x)$ for $p = 2$.

Note that $P(0) = 1$ and by (2.3) $Q^{(m)}(0) = 0$. Thus $\hat{\psi}^{(m)}(0) = 0$, for $0 \leq m \leq p-1$, and (2.11) is proved for $j = 0$ and $k = 1 - p$. The general case follows from a proper change of variables.

It can be shown that the polynomials $1, x, \ldots, x^{p-1}$ are in the subspace spanned by $\{\psi_{-1,k}\}$.

For example, by (2.3), (2.6) and induction, one can show that

$$\sum_k \phi(x-k) = 1. \tag{2.12}$$

Given any integer $J \ge 0$, we define

$$V_J = \operatorname{span}\{\psi_{jk} \mid -1 \le j < J,\ k \in \mathbb{Z}\}.$$

Obviously $V_J \subset V_{J+1}$. By PROPOSITION 2.1 $\cup_0^\infty V_J$ is dense in $L^2(\mathbb{R})$. It is worth noting that there is another interesting orthonormal basis for V_J in addition to $\{\psi_{jk}\}$. To see this, let

$$\phi_{jk}(x) = \sqrt{2^j}\phi(2^j x - k)$$

and

$$\tilde{V}_J = \operatorname{span}\{\phi_{Jk} \mid k \in \mathbb{Z}\}.$$

By (2.7), $\psi_{jk} \in \tilde{V}_{j+1}$. But (2.5) implies $\tilde{V}_j \subset \tilde{V}_{j+1}$. Thus we conclude $V_J \subset \tilde{V}_J$. Similarly the reverse inclusion holds. Therefore $V_J = \tilde{V}_J$, for all $J \ge 0$. Since $(\phi(x), \phi(x-k)) = \delta_{0k}$, $(\phi_{j\ell}, \phi_{jk}) = \delta_{\ell k}$. Consequently $\{\phi_{Jk}\}$ is another set of orthonormal basis of V_J. The sequence $\cdots \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \cdots$ defines the so called *multiresolution analysis* of $L^2(\mathbb{R})$ [20, 23]; while $\operatorname{span}\{\sqrt{2^j}\psi(2^j x - k) \mid k \in \mathbb{Z}\}$ is the orthogonal complement of $\tilde{V}_j$ in $\tilde{V}_{j+1}$.

For convenience of exposition, we shall use the term "frame" for a "basis" of a vector space in a weak sense. See [9, 16].

DEFINITION 2.2. *Let $\{\phi_n\}_{n=1}^\infty$ be a subset of a Banach space $(X, \|\cdot\|_X)$. Let $\operatorname{span}\{\phi_n \mid 1 \le n < \infty\}$ be the set of all elements $\sum \alpha_n \phi_n$ ($\alpha_n \in \mathbb{R}$) which converge (strongly) in X. Then $\{\phi_n\}$ is said to be a frame of X if* $\operatorname{span}\{\phi_n\} = X$.

Note that a frame is not necessarily an algebraic basis; the linear independence is not required. The main properties of wavelets used in this paper are summarized in the following proposition.

In the forthcoming sections, we shall be concerned with the space $L^2(\Omega)$, where Ω is a bounded open interval. It is clear that $\psi_{jk}|_\Omega$ form a frame for $L^2(\Omega)$. In fact, those with $\operatorname{supp}\psi_{jk} \cap \Omega \ne \emptyset$ will be sufficient for a frame. That is, $\{\psi_{jk} \mid j \ge -1,\ k \in I_j\}$, where

$$I_j = \{k \in \mathbb{Z} \mid 1 - R \le k \le 2^{\hat{j}}R - 1\}, \quad \text{with } \hat{j} = \max\{0, j\}. \tag{2.13}$$

Correspondingly, we define

$$\begin{aligned} V_J(\Omega) &= \operatorname{span}\{\psi_{jk}|_\Omega \mid -1 \le j < J,\ k \in I_j\}, \qquad \text{and} \\ \tilde{V}_J(\Omega) &= \operatorname{span}\{\phi_{Jk}|_\Omega \mid k \in I_J\}. \end{aligned} \tag{2.14}$$

It is easy to see that $\tilde{V}_J(\Omega) = V_J(\Omega)$. Also, $V_j(\Omega) \subset V_{j+1}(\Omega)$ and $\cup_0^\infty V_J(\Omega)$ is dense in $L^2(\Omega)$.

3. Anti-derivatives of wavelets and their approximation properties. In this section, the anti-derivatives of wavelets will be used to construct frames for $H_0^1(a,b)$, $H_\star^1(a,b)$ and $H^1(a,b)$ respectively. Approximation properties of these anti-derivatives will be discussed.

Given any $f \in L^1(a,b)$, we shall use the notation $\bar{f}$ to denote its mean value, i.e.

$$\bar{f} = \frac{1}{b-a}\int_a^b f\,dx.$$

We begin our discussion with general functions.

LEMMA 3.1. *Let* $\{\phi_n(x)\}_{n=1}^{\infty}$ *be a frame of* $L^2(a,b)$. *Then*

$$\left\{\Phi_n(x) = \int_a^x \phi_n \, ds - (x-a)\bar{\phi}_n \mid a \le x \le b\right\} \subset H_0^1(a,b), \tag{3.1}$$

$$\left\{\Phi_n(x) = \int_a^x \phi_n \, ds \mid a \le x \le b\right\} \subset H_\star^1(a,b), \tag{3.2}$$

and, for any $\alpha \in [-\infty, a)$,

$$\left\{\Phi_n(x) = \int_\alpha^x \phi_n \, ds \mid a \le x \le b\right\} \subset H^1(a,b) \tag{3.3}$$

are frames of $H_0^1(a,b)$, $H_\star^1(a,b)$ *and* $H^1(a,b)$ *respectively.*

Proof. We shall only present the proof for $H_0^1(a,b)$ since the proofs for other cases are similar. Let Φ_n be defined by (3.1), clearly $\Phi_n \in H_0^1(a,b)$. Given any $v \in H_0^1(a,b)$, let $w = v'$. Note that $\int_a^b w(x)\,dx = 0$.

Since $w \in L^2(a,b)$, there are numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that

$$\lim_{N\to\infty} \int_a^b (w - w_N)^2 \, dx = 0,$$

where $w_N(x) = \sum_{n=1}^N \alpha_n \phi_n$. Since $\bar{w}_N = \sum_{n=1}^N \alpha_n \bar{\phi}_n$, we have

$$\begin{aligned} |\bar{w}_N| &= \left|\frac{1}{b-a}\int_a^b w_N(x)\,dx\right| \\ &= \left|\frac{1}{b-a}\int_a^b [w_N(x) - w(x)]\,dx\right| \\ &\le \frac{1}{\sqrt{b-a}}\|w - w_N\|_0. \end{aligned}$$

Hence $\lim_{N\to\infty} \bar{w}_N = 0$ and

$$\|v' - \sum_{n=1}^N \alpha_n(\phi_n(x) - \bar{\phi}_n)\|_0 \le \|v' - w_N\|_0 + \sqrt{b-a}\,|\bar{w}_N| \to 0$$

as $N \to \infty$. Thus if we take $v_N = \sum_{n=1}^N \alpha_n \Phi_n$, then $v_N \in H_0^1(a,b)$ and

$$\lim_{N\to\infty} |v - v_N|_{H^1(a,b)} = 0.$$

As $|\cdot|_{H^1(a,b)}$ is a norm on $H_0^1(a,b)$, the proof is then completed. ∎

Recall that the wavelets $\{\psi_{jk} \mid j \geq -1 \text{ and } k \in I_j\}$ form a frame of $L^2(\Omega)$, hence the foregoing lemma can be applied. From here on, let $p \geq 1$, $R = 2p-1$, $\Omega = (0, R)$ and ψ_{jk} be the wavelets of order p. The following lemma is a direct consequence of LEMMA 3.1.

LEMMA 3.2. *For $j \geq -1$ and $k \in I_j$, the functions*

$$\Psi_{jk}(x) = \int_0^x \psi_{jk}\, ds - x\bar{\psi}_{jk}, \quad \text{for } 0 \leq x \leq R, \tag{3.4}$$

form a frame of $H_0^1(\Omega)$, and the functions

$$\Psi_{jk}(x) = \int_0^x \psi_{jk}\, ds, \quad \text{for } 0 \leq x \leq R. \tag{3.5}$$

form a frame of $H_\star^1(\Omega)$.

Apply LEMMA 3.1 with $\alpha = -R$, the functions

$$\Psi_{jk}(x) = \int_{-R}^x \psi_{jk}\, ds, \quad j \geq -1,\ 0 \leq x \leq R,$$

with $\mathrm{supp}\psi_{jk} \cap (-R, R) \neq \emptyset$, form a frame of $H^1(\Omega)$. We further observe that, for $0 \leq x \leq R$,

$$\begin{cases} \Psi_{jk}(x) = \int_{-\infty}^x \psi_{jk}\, ds, & \text{for } k \in I_j; \\ \Psi_{jk}(x) = \int_{-R}^\infty \psi_{jk}\, ds = \text{ a constant,} & \text{otherwise.} \end{cases}$$

So if we define

$$\Phi(x) = \int_{-\infty}^x \phi(s)\, ds, \quad \Psi(x) = \int_{-\infty}^x \psi(s)\, ds, \quad \text{for } 0 \leq x \leq R. \tag{3.6}$$

Then for $j \geq 0$, $\int_{-\infty}^x \psi_{jk}(s)\, ds = (1/\sqrt{2^j})\Psi(2^j x - k)$. Hence we have a frame for $H^1(\Omega)$ consisting of anti-derivatives of wavelets with $\mathrm{supp}\psi_{jk} \cap (0, R) \neq \emptyset$, as seen in the next lemma.

LEMMA 3.3. *$\{1, \Psi_{jk} \mid j \geq -1,\ k \in I_j\}$ is a frame of $H^1(\Omega)$ where*

$$\Psi_{jk}(x) = \begin{cases} \Phi(x-k), & \text{for } j = -1, \\ \frac{1}{\sqrt{2^j}}\Psi(2^j x - k), & \text{for } j \geq 0, \end{cases} \quad k \in I_j,\ 0 \leq x \leq R. \tag{3.7}$$

Note that $\Phi \in C^0$ for $p = 1$ and $\Phi \in C^1$ for $p = 2$. The support of $\Psi(x)$ is also $[0, R]$. But $\Phi(x) = 1$ for $x \in [R, \infty)$. Pictures of Φ and Ψ for $p = 2$ are shown in Figure 4.

Thus, from the anti-derivatives of wavelets, we have constructed a frame for each of the spaces $H_0^1(\Omega)$, $H_\star^1(\Omega)$ and $H^1(\Omega)$. We shall now discuss the approximation properties of these frames.

LEMMA 3.4. *For $j \geq 0$ and $p \geq 1$, let $f \in H^p$ and α_{jk} be given by (2.10). Then*

$$|\alpha_{jk}| \lesssim 2^{-js}|f|_{s,S_{jk}}, \quad 0 \leq s \leq p,$$

where $S_{jk} = \mathrm{supp}\psi_{jk}$.

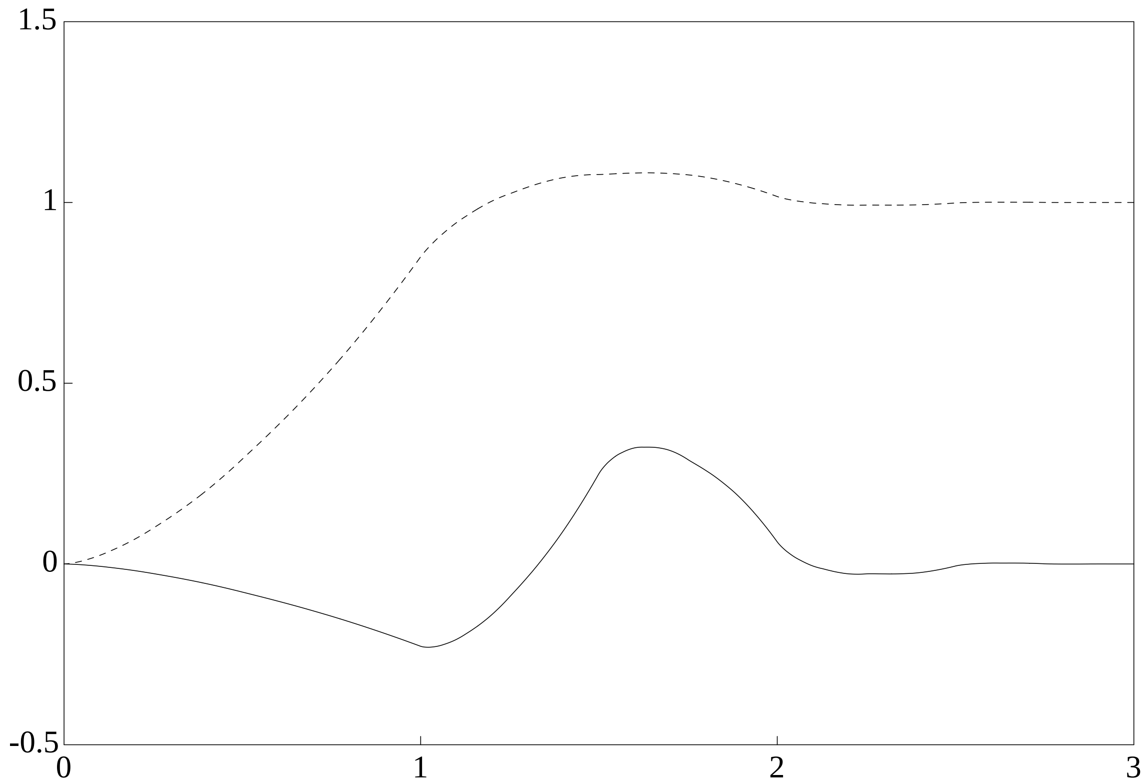

Figure 4. - - - $\Phi(x)$; —— $\Psi(x)$.

Proof. By (2.10) and (2.11), we have, for any polynomial $q(x)$ of degree $\le s-1$,

$$\alpha_{jk} = \int_{-\infty}^{\infty} (f-q)\psi_{jk}\,dx = \int_{S_{jk}} (f-q)\psi_{jk}\,dx$$
$$\le \|f-q\|_{0,S_{jk}} \|\psi_{jk}\| = \|f-q\|_{0,S_{jk}}.$$

Since $|S_{jk}| = 2^{-j}R$, a standard Bramble-Hilbert scaling argument shows that

$$|\alpha_{jk}| \le \inf_q \|f-q\|_{0,S_{jk}} \lesssim 2^{-js}|f|_{s,S_{jk}}. \qquad \blacksquare$$

A direct consequence of LEMMA 3.4 is

$$\|u - \sum_{j=-1}^{J} \sum_k \alpha_{jk}\psi_{jk}\| \lesssim 2^{-Js}|u|_s, \quad 0 \le s \le p.$$

But this estimate will not be directly used in this paper.

Since $\cup_0^\infty V_J(\Omega)$ is dense in $L^2(\Omega)$ and each $V_J(\Omega)$ is finite dimensional, it is natural to construct finite dimensional subspaces of $H_0^1(\Omega)$, etc., by the frame of $V_J(\Omega)$. The following theorems concern of the *distance* between these subspaces and $H_0^1(\Omega)$, etc.

THEOREM 3.1. *Given $J \ge 0$, define*

$$S_J = \operatorname{span}\Big\{\Psi_{jk} \mid -1 \le j < J,\ k \in I_j\Big\}. \tag{3.8}$$

Then S_J is a finite dimensional subspace of $H_0^1(\Omega)$ (resp. $H_\star^1(\Omega)$) if Ψ_{jk} are chosen from (3.4) (resp. (3.5)). For any $v \in H_0^1(\Omega) \cap H^{s+1}(\Omega)$ (resp. $v \in H_\star^1(\Omega) \cap H^{s+1}(\Omega)$), we have

$$\inf_{\chi \in S_J} |v - \chi|_{1,\Omega} \lesssim h^s |v|_{s+1,\Omega}, \quad 0 \le s \le p, \tag{3.9}$$

where $h = 2^{-J}$.

Proof. Without loss of generality, we will only present the proof for the case that $s = p$ and that Ψ_{jk} are given by (3.4). The proof for the other cases are similar.

Let $v \in H_0^1(\Omega) \cap H^{p+1}(\Omega)$. By the Sobolev extension theorem, there is a $\tilde{v} \in H^{p+1}(\mathbb{R})$ such that $\tilde{v}(x) = v(x)$ for $x \in \Omega$ and

$$\|\tilde{v}\|_{p+1} \lesssim \|v\|_{p+1,\Omega}.$$

By Lemma 3.1 and (2.10),

$$\lim_{\ell \to \infty} |v - \sum_{j=-1}^{\ell} \sum_{k \in I_j} \alpha_{jk} \Psi_{jk}|_{1,\Omega} = 0$$

where $\alpha_{jk} = (\tilde{v}', \psi_{jk})$. Let $e_j = \sum_{k \in I_j} \alpha_{jk} \Psi_{jk}$ and $\chi = \sum_{j=-1}^{J-1} e_j$. Then $\chi \in S_J$ and

$$|v - \chi|_{1,\Omega} \lesssim \sum_{j \ge J} \|e_j'\|_{0,\Omega}.$$

For each j, let $I_j' \subset I_j$ consist of the indices k such that $\mathrm{supp}\psi_{jk} \subset [0, R]$. That is, $0 \le k \le (2^j - 1)R$. Let $I_j'' = I_j \setminus I_j'$. Since $j \ge 0$, we have

$$\begin{aligned}
\int_0^R |e_j'|^2 \, dx &= \sum_{k \in I_j} \sum_{\ell \in I_j} \alpha_{jk} \alpha_{j\ell} \int_\Omega (\psi_{jk} - \bar{\psi}_{jk})(\psi_{j\ell} - \bar{\psi}_{j\ell}) \, dx \\
&= \sum_{k \in I_j'} \alpha_{jk}^2 + 2 \sum_{\substack{k \in I_j' \\ \ell \in I_j''}} \alpha_{jk} \alpha_{j\ell} \int_\Omega \psi_{jk} (\psi_{j\ell} - \bar{\psi}_{j\ell}) \, dx \\
&\quad + \sum_{k,\ell \in I_j''} \alpha_{jk} \alpha_{j\ell} \int_\Omega (\psi_{jk} - \bar{\psi}_{jk})(\psi_{j\ell} - \bar{\psi}_{j\ell}) \, dx.
\end{aligned}$$

Note that if $k \in I_j'$ then $\int_\Omega \psi \, dx = \int_{-\infty}^\infty \psi \, dx$ and $\bar{\psi}_{jk} = 0$. Also note that if $k \in I_j'$ and $\ell \in I_j''$, then $k \ne \ell$ and $\mathrm{supp}\psi_{jk} \subset \Omega$. Thus it follows from (2.9) and (2.11) that

$$\sum_{\substack{k \in I_j' \\ \ell \in I_j''}} \alpha_{jk} \alpha_{j\ell} \int_\Omega \psi_{jk} (\psi_{j\ell} - \bar{\psi}_{j\ell}) \, dx = \sum_{\substack{k \in I_j' \\ \ell \in I_j''}} \alpha_{jk} \alpha_{j\ell} \int_{-\infty}^\infty \psi_{jk} \psi_{j\ell} \, dx = 0.$$

Since $| \int_\Omega \psi_{jk} \, dx| \le \sqrt{R} \, \|\psi_{jk}\|_{0,\Omega} \le \sqrt{R}$, thus

$$\|\psi_{jk} - \bar{\psi}_{jk}\|_{0,\Omega} \le \|\psi_{jk}\| + \sqrt{R} \, |\bar{\psi}_{jk}| \le 2.$$

Also, since $|I_j''| = 2R - 2$,

$$4 \sum_{k,\ell \in I_j''} |\alpha_{jk} \alpha_{j\ell}| \le 2 \sum_{k,\ell \in I_j''} (\alpha_{jk}^2 + \alpha_{j\ell}^2) = (4R - 2) \sum_{k \in I_j''} \alpha_{jk}^2.$$

Therefore

$$\int_0^R |e_j'|^2\,dx \le \sum_{k\in I_j'} \alpha_{jk}^2 + 4\sum_{k,\ell\in I_j''} |\alpha_{jk}\alpha_{j\ell}| \le (4R-1)\sum_{k\in I_j}\alpha_{jk}^2.$$

By LEMMA 3.4,

$$|\alpha_{jk}| \lesssim 2^{-jp}|\tilde v'|_{p,S_{jk}}.$$

Thus

$$\sum_{k\in I_j}\alpha_{jk}^2 \lesssim 2^{-2jp}\sum_{k\in I_j}|\tilde v|_{p+1,S_{jk}}^2.$$

We note that, for each $k\in I_j$, there are at most $R-1$ indices $\ell > k$ such that $\mathrm{supp}\,\psi_{j\ell}$ has a nonzero intersection with $\mathrm{supp}\,\psi_{jk}$. Therefore

$$\sum_{k\in I_j}\alpha_{jk}^2 \lesssim 2^{-2jp}|\tilde v|_{p+1}^2 \lesssim 2^{-2jp}\|v\|_{p+1,\Omega}^2.$$

Consequently,

$$|v-\chi|_{1,\Omega} \lesssim \|v\|_{p+1,\Omega}\sum_{j\ge J}2^{-jp} \le 2h^p\,\|v\|_{p+1,\Omega}. \tag{3.10}$$

Now let v_p be the polynomial of degree $\le p$ such that

$$v_p(\frac{i}{p}R) = v(\frac{i}{p}R), \quad 0\le i\le p.$$

Then a standard Bramble-Hilbert argument shows that $\|v-v_p\|_{p+1,\Omega} \lesssim |v|_{p+1,\Omega}$. Applying (3.10) with $v-v_p$ in place of v, we get

$$|v-(v_p+\chi)|_{1,\Omega} \lesssim h^p|v|_{p+1,\Omega}.$$

By (3.4), it can be shown that $v_p\in S_J$, the desired result then follows. ∎

THEOREM 3.2. *Let Ψ_{jk} be given by (3.7) and $J\ge 0$. Define*

$$S_J = \mathrm{span}\left\{1,\ \Psi_{jk}\mid -1\le j<J,\ k\in I_j\right\}.$$

Then S_J is a finite dimensional subspace of $H^1(\Omega)$ and, for all $v\in H^{s+1}(\Omega)$, the estimate (3.9) holds.

Before the end of this section, we would like to remark that we can also construct frames of S_J by the frame of $\tilde V_J(\Omega)$. Namely, for subspaces of $H_0^1(\Omega)$ or $H_\star^1(\Omega)$, take

$$\Phi_{jk}(x) = \int_0^x \phi_{jk}\,ds - x\bar\phi_{jk}, \quad \text{for } 0\le x\le R, \tag{3.11}$$

or

$$\Phi_{jk}(x) = \int_0^x \phi_{jk}\,ds, \quad \text{for } 0\le x\le R, \tag{3.12}$$

respectively, then $S_J = \mathrm{span}\{\Phi_{Jk}\mid k\in I_J\}$. For the subspace S_J of $H^1(\Omega)$, take

$$\Phi_{jk}(x) = \frac{1}{\sqrt{2^j}}\Phi(2^jx-k)|_\Omega \tag{3.13}$$

and $S_J = \mathrm{span}\{1, \Phi_{Jk}\mid k\in I_J\}$.

4. Applications to two-point boundary value problems. In this section, we apply the wavelets of order p to numerical solutions of a two-point boundary value problem*

$$-(q(x)u')' + r(x)u = f(x), \quad \text{for } x \in (0, R), \tag{4.1}$$

with either one of the boundary conditions:

$$\begin{cases} u(0) = u(R) = 0, & \text{(Dirichlet condition)} \\ u(0) = u'(R) = 0, & \text{(mixed condition)} \\ u'(0) = u'(R) = 0. & \text{(Neumann condition)} \end{cases}$$

We also let $\Omega = (0, R)$, where $R = 2p - 1$. We assume that $f \in L^2(\Omega)$, the coefficients q and r are smooth in $[0, R]$ with

$$0 < \underline{q} \leq q(x) \leq \overline{q} \ \text{ and } \ 0 \leq r(x) \leq \overline{r}. \tag{4.2}$$

We now define Galerkin-wavelets methods for solving (4.1). According to the boundary condition, we construct a finite dimensional subspace S_J of the solution space as defined in Theorem 3.1 and 3.2. Then we solve numerically the Galerkin projection of the solution in S_J. We first give a general account of the method and the error estimates in 4.1, then construct two bases for each S_J in 4.2. Some computational aspects are considered in 4.3 and 4.4. Finally, we discuss connections with hierarchical bases in the last subsection.

4.1. Error estimates. In this subsection, we shall only consider the Dirichlet boundary value problem. Thus S_J, $J \geq 0$, is the subspace of $H_0^1(\Omega)$ with frames Ψ_{jk} from (3.4) or Φ_{Jk} from (3.11). The other cases are similar with an appropriate choice of Ψ_{jk} or Φ_{Jk}. The variational form of (4.1) is

$$a(u, v) = (f, v), \quad \text{for all } v \in H_0^1(\Omega), \tag{4.3}$$

where $a(\cdot, \cdot)$ is the bilinear form defined by

$$a(u, v) = \int_0^R qu'v' + ruv \, dx. \tag{4.4}$$

Clearly $a(\cdot, \cdot)$ is continuous and coercive on $H_0^1(\Omega)$. It is well-known that, by Lax-Milgram lemma (cf. [3]), there is a unique weak solution $u \in H_0^1(\Omega)$ for (4.3). Let u_J be the Galerkin projection of u on S_J defined by

$$a(u_J, \chi) = (f, \chi), \quad \text{for all } \chi \in S_J. \tag{4.5}$$

It is also clear that (4.5) has a unique solution such that

$$\|u - u_J\|_{1,\Omega} \lesssim \inf_{\chi \in S_J} \|u - \chi\|_{1,\Omega}.$$

By Theorem 3.1 and a standard duality argument (cf. [3]), the error estimates for both H^1 and L^2 norms can be obtained.

Theorem 4.1. *Let u and u_J be the solutions of* (4.3) *and* (4.5) *respectively. Then for any $u \in H_0^1(\Omega) \cap H^{s+1}(\Omega)$,*

$$\|u - u_J\|_{0,\Omega} + h\|u - u_J\|_{1,\Omega} \lesssim h^{s+1}\|u\|_{s+1,\Omega}, \quad 0 \leq s \leq p, \tag{4.6}$$

where $h = 2^{-J}$.

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* Without loss of generality, a special interval $(0, R)$ is used in this model. More general interval (a, b) can be reduced to $(0, R)$ by a simple change of variable.

4.2. Basis functions from frames. It is well-known that the approximate solution u_J can be obtained by solving a linear algebraic system by means of a basis of S_J. But the frames of S_J discussed earlier are not bases. In order to obtain a basis from a frame, we need to exclude some linearly dependent functions. We first study the bases for $V_J(\Omega)$.

PROPOSITION 4.2. *For any $J \geq 0$, $\dim V_J(\Omega) = 2^J R + R - 1$. Define the index set D_j such that*

$$k \in D_j \quad \Longleftrightarrow \quad \begin{cases} 1 - R \leq k \leq R - 1, & \text{if } j = -1; \\ p - R \leq k \leq 2^j R - p, & \text{if } j \geq 0. \end{cases} \tag{4.7}$$

Then

$$\{\phi_{Jk} \mid 1 - R \leq k \leq 2^J R - 1\} \tag{4.8}$$

and

$$\{\psi_{jk} \mid -1 \leq j < J,\ k \in D_j\} \tag{4.9}$$

are bases of $V_J(\Omega)$.

Proof. For any $J \geq 0$, the functions in (4.8) form a frame of $V_J(\Omega)$, as defined in (2.14). To show they actually form a basis, it suffices to show that they are linearly independent. Now suppose

$$\chi = \sum_{k=1-R}^{2^J R - 1} \alpha_k \phi_{Jk} = 0 \ \text{ in } \Omega.$$

Note that for any $0 \leq m \leq 2^J R - R$, the support of ϕ_{Jm} lies entirely in $[0, R]$, thus $(\phi_{Jm}, \phi_{J\ell}) = \delta_{m\ell}$, for all ℓ. Thus $\alpha_{Jm} = 0$. Observe that the value of $\chi(x)$ for $R-2 \leq 2^j x \leq R-1$ is solely determined by $\alpha_{-1}\phi_{J,-1}$. But $\phi_{J,-1} \neq 0$ in that interval, so $\alpha_{J,-1} = 0$. The same argument shows that $\alpha_k = 0$ for all $1 - R \leq k \leq 2^J R - 1$. Consequently the functions in (4.8) form a basis of $V_J(\Omega)$. Using this basis, we count that the dimension of $V_J(\Omega)$ is $2^J R + R - 1$.

Now we show by induction that the wavelets in (4.9) are also basis functions of $V_J(\Omega)$. For $J = 0$, the functions in (4.9) are precisely those in (4.8), so they form a basis for $V_0(\Omega)$. Suppose that ψ_{jk}, for $0 \leq j \leq J - 1$ and $k \in D_j$, form a basis of $V_J(\Omega)$. We shall now prove that ψ_{jk}, for $0 \leq j \leq J$ and $k \in D_j$, form a basis of $V_{J+1}(\Omega)$. To see this, we first prove that they are linearly independent. Assume

$$\chi = \sum_{j=1}^{J} \sum_{k \in D_j} \alpha_{jk} \psi_{jk} = 0 \ \text{ in } \Omega.$$

Then there are uniquely determined numbers β_k such that

$$\chi = \sum_{k=1-R}^{2^J R - 1} \beta_k \phi_{Jk} + \sum_{k \in D_J} \alpha_{Jk} \psi_{Jk} \ \text{ in } \Omega.$$

For the same reason as we put in the first paragraph, we have $\alpha_{Jk} = 0$ for all $0 \leq k \leq 2^J R - R$. By (2.7),

$$\psi(2^J x - k) = -c_0 \phi(2^{J+1} x - (R + 2k)) + \cdots + c_R \phi(2^{J+1} x - 2k).$$

Hence for any $p - R \leq k < 0$ and $2^J R - R < k \leq 2^J R - p$, we have $1 \leq R + 2k < R$ and $2^{J+1} R - 2R \leq 2k \leq 2^{J+1} R - R - 1$ respectively. That means either $\phi(2^{J+1} x - (R + 2k))$ or $\phi(2^{J+1} x - 2k)$ has the support in $[0, R]$. The orthogonality of $\phi(x)$ and $\phi(2x)$ in $L^2(\mathbb{R})$ implies that either $\alpha_{Jk} c_0 = 0$ or $\alpha_{Jk} c_R = 0$. Since $c_0 \neq 0$ and $c_R \neq 0$, we conclude $\alpha_{Jk} = 0$. Thus $\alpha_{Jk} = 0$ for all $k \in D_J$. By the induction hypothesis, $\alpha_{jk} = 0$ for all $0 \leq j \leq J$ and $k \in D_j$.

Therefore we have shown that the set $\{\psi_{jk} \mid -1 \leq j \leq J,\ k \in D_j\}$ is linearly independent in $V_{J+1}(\Omega)$. Counting the cardinality, we see it is a basis for $V_{J+1}(\Omega)$. The general result follows by induction. ∎

If (4.1) is imposed with the Neumann boundary condition, and $S_J \subset H^1(\Omega)$ is the space defined in THEOREM 3.2, then by the linear independence of (4.8) and (4.9), it is easy to see that

$$\{1, \Psi_{jk} \mid -1 \le j < J,\ k \in D_j\},\ \text{where } \Psi_{jk} \text{ is defined in (3.7)} \tag{4.10}$$

and

$$\{1, \Phi_{Jk} \mid k \in I_J\},\ \text{where } \Phi_{jk} \text{ is defined in (3.13)} \tag{4.11}$$

are bases of S_J.

The bases for $S_J \subset H^1_*(\Omega)$ are also easy to construct if (4.1) is imposed with the mixed boundary condition. But there is another twist for the subspace $S_J \subset H^1_0(\Omega)$, because of the extra term $x\bar{\psi}_{jk}$. Let us apply (3.4) to the functions in (4.9), with $j = -1$. By (2.12), we have

$$\begin{aligned}\sum_{k=1-R}^{R-1} \psi_{-1,k} &= \sum_{k=1-R}^{R-1} \Big(\int_0^x \phi(s-k)\,ds - \frac{x}{R}\int_0^R \phi(x-k)\,dx \Big) \\ &= \int_0^x \sum_k \phi(s-k)\,ds - \frac{x}{R}\int_0^R \sum_k \phi(x-k)\,dx \\ &= \Big(x - \frac{x}{R}R\Big) = 0.\end{aligned} \tag{4.12}$$

Thus $\Psi_{-1,k}$, for $k \in D_{-1}$, are linearly dependent. The reason is that 1 can be spanned by $\phi(x-k)$.

PROPOSITION 4.3. *Let D_j be the index set defined in (4.7). Let $\tilde{D}_j = D_j$ for $j \ge 0$ and $\tilde{D}_{-1} = D_{-1} \setminus \{1-R\}$. Then*

$$\{\Psi_{jk} \mid -1 \le j < J,\ k \in \tilde{D}_j\},\ \textit{where } \Psi_{jk} \textit{ is defined in (3.4)} \tag{4.13}$$

and

$$\{\Phi_{Jk} \mid 2-R \le k \le 2^J R - 1\},\ \textit{where } \Phi_{jk} \textit{ is defined in (3.11)} \tag{4.14}$$

are bases of S_J, where S_J is the subspace of $H^1_0(\Omega)$ defined in THEOREM 3.1.

Proof. Let Ψ_{jk} be the functions in (4.13). Suppose $\sum_{jk} \alpha_{jk}\Psi_{jk} = 0$, which can be written as

$$\int_0^x \chi\,ds = \bar{\chi}x \quad \text{in } \Omega, \tag{4.15}$$

where

$$\chi = \sum_{j=-1}^{J-1} \sum_{k \in \tilde{D}_j} \alpha_{jk}\psi_{jk}.$$

Obviously (4.15) implies that $\chi = \bar{\chi}$ is a constant function in Ω. By (2.12), we have

$$-\bar{\chi}\psi_{-1,1-R} + \sum_{k \in \tilde{D}_{-1}} (\alpha_{-1,k} - \bar{\chi})\psi_{-1,k} + \sum_{j=0}^{J-1} \sum_{k \in D_j} \alpha_{jk}\psi_{jk} = 0.$$

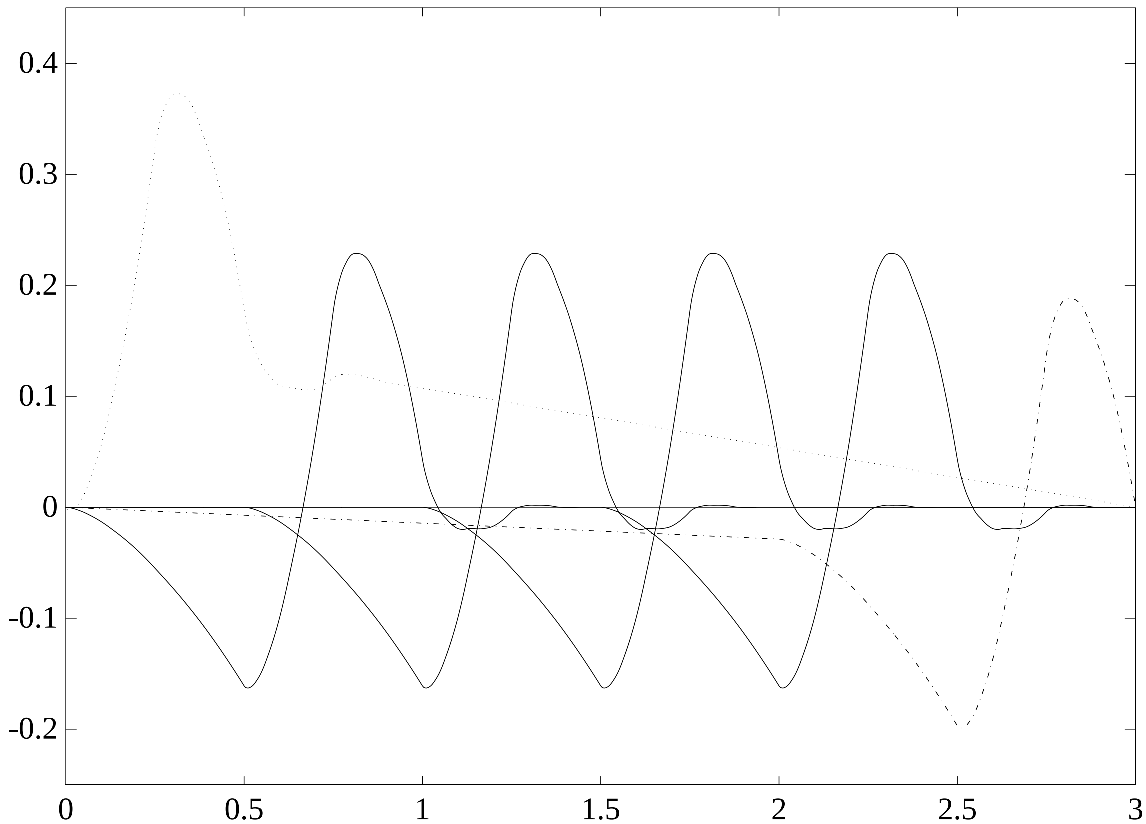

Figure 5. $\cdots\ \Psi_{1,-1}$; —— Ψ_{1k}, $0 \le k \le 3$; $\cdot - \cdot$ Ψ_{14}.

Since ψ_{jk} in (4.9) are linearly independent, so $\bar{\chi} = 0$ and $\alpha_{jk} = 0$. Hence the functions in (4.13) are linearly independent.

We have already seen in (4.12) that $\Psi_{-1,1-R}$ is a linear combination of $\Psi_{-1,k}$, for $k \in \tilde{D}_{-1}$. But, as defined in Theorem 3.1, S_J is spanned by Ψ_{jk} for $-1 \le j < J$ and $k \in D_j$. Therefore, the wavelets in (4.13) can span S_J, and thus form a basis.

Similarly we can show that the set of functions in (4.14) is linearly independent. Since its cardinality is equal to the dimension of S_J, it is a basis. ∎

Pictures of Ψ_{1k} for $k \in \tilde{D}_1$ are shown in Figure 5.

4.3. Solution of linear algebraic systems. Suppose that (4.1) is imposed with Dirichlet boundary condition and the basis of S_J is given by (4.13). Let $u_J = \sum u_{jk}\Psi_{jk}$ be the Galerkin projection of u in S_J, the coefficients u_{jk} are then determined by the following linear system of equations.

$$\sum_{j,k} u_{jk}\, a(\Psi_{jk}, \Psi_{\ell m}) = (f, \Psi_{\ell m}), \quad \text{for all } -1 \le \ell < J,\ m \in \tilde{D}_\ell.$$

We now label the double indices linearly. Let $\iota : (j,k) \mapsto i$ be the labelling function which enumerate (j,k) lexicographically. That is, $\iota(j,k) \le \iota(\ell,m)$ if $j \le \ell$ or $j = \ell$ and $k \le m$. We then get a linear system of the form $A\boldsymbol{u} = \boldsymbol{f}$, where the matrix $A = (a_{ij})$ and the vector $\boldsymbol{f} = (f_j)$ have the entries

$$a_{ij} = a(\Psi_i, \Psi_j), \quad f_j = (f, \Psi_j). \tag{4.16}$$

Here $a(\cdot,\cdot)$ is the bilinear form defined in (4.4) and A corresponds to the stiffness matrix in the usual finite element method. Because Ψ_i form a basis and $a(\cdot,\cdot)$ is symmetric and coercive on S_J, A is symmetric and positive definite.

If $r(x) = 0$ and $q(x) = 1$, then A is sparse. In fact, let P be the permutation matrix which relabels Ψ_{jk} in such a way that those with $0 \le k \le 2^j R - R$ (which supports lie entirely in $[0, R]$) are indexed together after the others, one can see that

$$A = P \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} P^t. \tag{4.17}$$

Let $N = \dim S_J$, then B is a full square matrix with order $O(\log N)$. As a result, the eigenvalues of A are mostly 1's, with a few exceptions caused by the wavelets near the boundary. Similar results also hold if $q(x)$ and $r(x)$ are not constant. In fact, the following proposition can be proved by a simple application of the Courant-Fischer Minimax Theorem [11].

PROPOSITION 4.4. *For general coefficients $q(x)$ and $r(x)$ satisfying* (4.2), *all but $O(\log N)$ of the eigenvalues of A are uniformly bounded below and above.*

If the conjugate gradient method is used to solve a linear system, one finds that if most of the eigenvalues are clustered in one place, the convergence speed can be fast even if the condition number is big.

PROPOSITION 4.5. ([1]) *Let $\boldsymbol{u}^k$ be the k-th approximated solution of $A\boldsymbol{u} = \boldsymbol{f}$ by the conjugate gradient method. Suppose the spectrum of $A = \sigma_0 \cup \sigma_1$. If $\ell = |\sigma_1|$ and*

$$\kappa = (\max_{\sigma_0} \lambda) \Big/ (\min_{\sigma_0} \lambda).$$

Then

$$\|\boldsymbol{u} - \boldsymbol{u}^k\|_A \le M \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{k-\ell},$$

where $\|\cdot\|_A^2 = (A\cdot, \cdot)$ and M is a constant depending on the spectrum of A.

By PROPOSITION 4.4 and 4.5, the convergence of the conjugate gradient method for solving $A\boldsymbol{u} = \boldsymbol{f}$ will be uniform after $O(\log N)$ steps. This is already very attractive since A is a matrix of order N.

To further improve the performance of the conjugate gradient method, we have a natural choice of preconditioner for the matrix A. Let B be the matrix in (4.17). Since B is small, it is easy to invert B by a direct method. Evidently B and B^{-1} are symmetric and positive definite, hence there exists the Cholesky decomposition, say $B^{-1} = LL^t$. Then SS^t is a good preconditioner where

$$S = P \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix}. \tag{4.18}$$

That is, instead of solving $A\boldsymbol{x} = \boldsymbol{b}$, we solve the equivalent problem

$$S^t A S \boldsymbol{y} = S^t \boldsymbol{b}, \quad \boldsymbol{x} = S\boldsymbol{y},$$

where $S^t A S$ is a well-conditioned matrix. The action of P can be easily derived during the computation, so we only have to compute and store the lower triangular matrix L which has $O(\log^2 N)$

entries. Theoretically the condition number of S^tAS is one if $q = 1$ and $r = 0$; in general it is bounded above by $\overline{q} + \overline{r}/\underline{q}$. The preconditioner S is made possible by the orthogonality property of wavelets.

If (4.1) is implosed with mixed or Neumann boundary conditions, then it is preferred to construct S_J by Φ_{Jk} in (3.12) or (3.13). We also denote the corresponding stiffness matrix by A. Since there are constantly many $(2R-2)$ basis functions which do not lie in $[0, R]$, all but $O(1)$ of A's eigenvalues are uniformly bounded. Finally, we remark that (1) the matrix A is sparse if $r = 0$, and (2) we can construct a preconditioner in the same manner as S in (4.18).

4.4. Reduction of operations. Let A be the stiffness matrix by means of the basis $\{\Psi_{jk}\}$ in (4.13). If q is not a constant or $r \neq 0$, then A is a full matrix and its multiplication with a vector becomes expensive. We can overcome this drawback by changing basis (as is done in hierarchical basis, cf [28]). Recall that Φ_{Jk} in (4.14) also form a basis for S_J. We label Φ_{Jk} by k and let $\hat{A}$ be the corresponding stiffness matrix.

Consider the bases $\{\psi_{jk}\}$ in (4.9) and $\{\phi_{Jk}\}$ in (4.8) for $V_J(\Omega)$. Let Q be the matrix of changing basis. That is,

$$(\psi_1\ \psi_2 \cdots \psi_N) = (\phi_1\ \phi_2 \cdots \phi_N)\, Q.$$

We shall show that Q has $O(N)$ nonzero entries. Since the construction from ψ_{jk} to Ψ_{jk} is linear, with little modifications we can change from $\{\Psi_{jk}\}$ to $\{\Phi_{Jk}\}$ in $O(N)$ operations.

To simplify the presentation and explain more clearly the main idea, we further ignore the coefficients $\sqrt{2^j}$. Then the problem reduces to finding the matrix Q such that

$$\begin{pmatrix} \phi(x+R-1) \\ \vdots \\ \psi(2^{J-1}x - 2^{J-1}R + p) \end{pmatrix} = Q^t \begin{pmatrix} \phi(2^J x - R + 1) \\ \vdots \\ \phi(2^J x - 2^J R + 1) \end{pmatrix}.$$

Proposition 4.6. *For any $\boldsymbol{x} \in \mathbb{R}^N$, $O(N)$ operations are sufficient for computing the multiplication $Q\boldsymbol{x}$.*

Proof. By (2.5) and (2.7), it is easy to see that

$$\begin{pmatrix} \phi(x+R-1) \\ \vdots \\ \phi(x-R+1) \\ \psi(x+R-p) \\ \vdots \\ \psi(x-R+p) \end{pmatrix} = \tilde{Q}_1^t \begin{pmatrix} \phi(2x+R-1) \\ \\ \vdots \\ \\ \psi(2x-2R+1) \end{pmatrix},$$

where $\tilde{Q}_1$ is a matrix of order $\dim V_1(\Omega)$ and each column of $\tilde{Q}_1$ consists of the coefficients c_k. Hence there are at most $2p$ nonzero elements on each column of $\tilde{Q}_1$. Inductively,

$$Q = Q_J \cdots Q_2 Q_1.$$

Here, for any $1 \leq j \leq J$,

$$Q_j = \begin{pmatrix} \tilde{Q}_j & 0 \\ 0 & I_j \end{pmatrix},$$

where $\tilde{Q}_j$ is a sparse matrix of order $\dim V_j(\Omega)$ and I_j is the identity matrix of order $\dim V_J(\Omega) - \dim V_j(\Omega)$. Since $\tilde{Q}_j$ has $O(2^j)$ nonzero entries, a multiplication $Q\boldsymbol{x}$ needs

$$O(1) + O(2) + \cdots + O(2^J) = O(2 \cdot 2^J) = O(N)$$

operations. ∎

Because of the term $x\bar{\phi}_{Jk}$, Φ_{Jk} are fully supported in $[0,R]$ and $\hat{A}$ is still full. This can be modified by a further change of basis. Consider the functions

$$\Theta_{Jk} = \Phi_{J,k-1} - \Phi_{Jk}, \quad \text{for } 2-R \le k \le 2^J R - 1. \tag{4.19}$$

It is easy to show that $\operatorname{span}\{\Theta_{Jk}\} \subset S_J$ and that they are linearly independent. Counting the dimension, they form another basis for S_J. Note that, for $0 \le k \le 2^J R - R$, $\bar{\phi}_{Jk}$ are the same constants and

$$\operatorname{supp}\Theta_{Jk} = \operatorname{supp}\Theta'_{Jk} = \operatorname{supp}\phi_{J,k-1} \cup \operatorname{supp}\phi_{Jk}.$$

So all but $2R-2$ of Θ_{Jk} and Θ'_{Jk} are locally supported in $[0,R]$. Let $\tilde{A}$ be the stiffness matrix corresponding to this basis, then $\tilde{A}$ has $O(N)$ nonzero entries.

Therefore, there is a linear transformation T such that

$$A = T^t \tilde{A} T.$$

For any $\boldsymbol{x} \in \mathbb{R}^N$, $O(N)$ operations are sufficient to compute a multiplication $T\boldsymbol{x}$. Since $\tilde{A}$ is sparse, we can reduce the number of operations for $A\boldsymbol{x}$ to $O(N)$.

If (4.1) is imposed with mixed or Neumann boundary conditions and S_J is constructed by $\{\Phi_{Jk}\}$. Then A is already sparse when $r = 0$. Otherwise A is full and we can change the basis to Θ_{Jk} in the same manner as in (4.19).

4.5. Relations with hierarchical bases in finite elements. As we have mentioned in the introduction, we now observe the connection between the wavelet basis (4.13) of the case $p = 1$ and the hierarchical basis [27] in one-dimensional piecewise linear finite element spaces. Let S_J be the basis of $H_0^1(\Omega)$ defined in THEOREM 3.1, where Ψ_{jk} are defined in (4.13). Since $p = 1$, now $R = 2p - 1 = 1$ and $\Omega = (0,1)$; so $\tilde{D}_{-1} = \emptyset$. For all $k \in D_j$, $j \ge 0$, we find $\operatorname{supp}\psi_{jk} \subset [0,1]$, so $\bar{\psi}_{jk} = 0$. It is easy to see that Ψ_{jk} is a hat function with support $[k/2^j, (k+1)/2^j]$ and height $1/2\sqrt{2^j}$. That is, the basis functions $\{\Psi_{jk}\}$ for S_J are the piecewise linear hierarchical basis functions with the uniform mesh size $h = 2^{-J}$.

On the other hand, consider Θ_{Jk} in (4.19). In this case $\bar{\phi}_{J,k-1} - \bar{\phi}_{Jk} = 0$. Observe that Θ_{Jk}, for $1 \le k \le 2^J - 1$, are the scaled piecewise linear nodal basis with the uniform mesh size $h = 2^{-J}$.

5. Numerical examples. In this section, we verify our claims on some test problems of the form (4.1), with Dirichlet or Neumann boundary conditions. We have seen in section 4.4 that the finite elements with $p = 1$ are exactly the hierarchical piecewise linear finite elements. So now we test the methods with wavelets of order 2. That is, $p = 2$ and $R = 3$. The programs are coded in `FORTRAN`. The tests are done on a `SUN SparkStation 1`.

We implement the functions ϕ and ψ by their values at dyadic points in $[0,R]$; that is, the values of $\phi(x)$ and $\psi(x)$ at $x = k/2^D$, for $0 \le k \le 2^D R$ (the functions vanish for k out of this range). The data are stored in disk files. Then we temporarily generate ϕ and ψ with resolution $D+1$, from which we compute the functions Φ and Ψ. The integrations are done by Simpson's numerical quadrature rule. Similarly we store the dyadic values of Φ and Ψ in $[0,R]$ with the common denominator 2^D. In practice, the choice of D will affect the accuracy of numerical quadratures. The highest level J which can yield accurate solution is also related with the resolution D. To choose D empirically, consider the problem

$$\begin{cases} -u'' = 1, & \text{for } x \in (0,3), \\ u(0) = u(3) = 0. \end{cases} \tag{5.1}$$

The solution $u = -\frac{1}{2}x^2 + \frac{3}{2}x$ just lies in the subspace S_0, if the wavelets are of order $p \geq 2$. Hence the errors are due to the computations. The error $\|u - u_J\|_{0,\Omega}$ is approximately 10^{-8} for $D = 8$, and 10^{-12} for $D = 12$. In most of the computations shown below, we set $D = 8$.

The function values of ϕ, etc., are repeatedly used during the computation, hence they are read by the program to save some running time. Note that $\Phi(x) = 1$ for $x \geq R$. This is the only function involved which does not have a compact support. The programs must treat this exception carefully.

We now test some Dirichlet boundary value problems. Let the finite element spaces S_J be spanned by basis Ψ_{jk} chosen from (4.13). As a first example, we modify (5.1) a little in order to observe the convergence order:

$$\text{(5.2)} \qquad \begin{cases} -u'' = 3x - 3, & \text{for } x \in (0,3), \\ u(0) = u(3) = 0. \end{cases}$$

The computational results are summarized in Table 1.

(5.2)	$h^{-3}\|u - u_J\|_{0,\Omega}$	$h^{-2}\lvert u - u_J\rvert_{1,\Omega}$	CG
$J = 0$	0.09153	0.4750	3
$J = 1$	0.08964	0.5307	5
$J = 2$	0.09051	0.5598	7
$J = 3$	0.09114	0.5766	8
$J = 4$	0.08994	0.5811	10

Table 1.

In Table 1 and all of the following tables, $h = 2^{-J}$ and the columns labelled "CG" and "PCG" list respectively the numbers of iterations needed for the conjugate gradient and preconditioned conjugate gradient methods to obtain the displayed error. For problem (5.2), the preconditioner SS^t is exactly A^{-1}. Thus we get exact solutions with one iteration.

In the second example, we made up a problem with transcendental coefficients:

$$\text{(5.3)} \qquad \begin{cases} -(e^{-x}u')' = \frac{\pi}{3}e^{-x}\Big(\cos(\frac{\pi}{3}x) + \frac{\pi}{3}\sin(\frac{\pi}{3}x)\Big), & \text{for } x \in (0,3), \\ u(0) = u(3) = 0. \end{cases}$$

The computational results are summarized in Table 2. We also compare the numbers of iterations by conjugate gradient method with or without the preconditioner.

(5.3)	$h^{-3}\|u - u_J\|_{0,\Omega}$	$h^{-2}\lvert u - u_J\rvert_{1,\Omega}$	CG	PCG
$J = 0$	0.02546	0.0980	3	3
$J = 1$	0.02622	0.1285	6	6
$J = 2$	0.02502	0.1436	12	9
$J = 3$	0.02509	0.1528	14	10
$J = 4$	0.02457	0.1651	16	10

Table 2.

In order to observe the trend of converging in a picture, we tried a problem with higher frequency solution, say $u(x) = \sin(\xi x)$, where $\xi = 4\pi/3$. It is not necessary to have a complicated $q(x)$

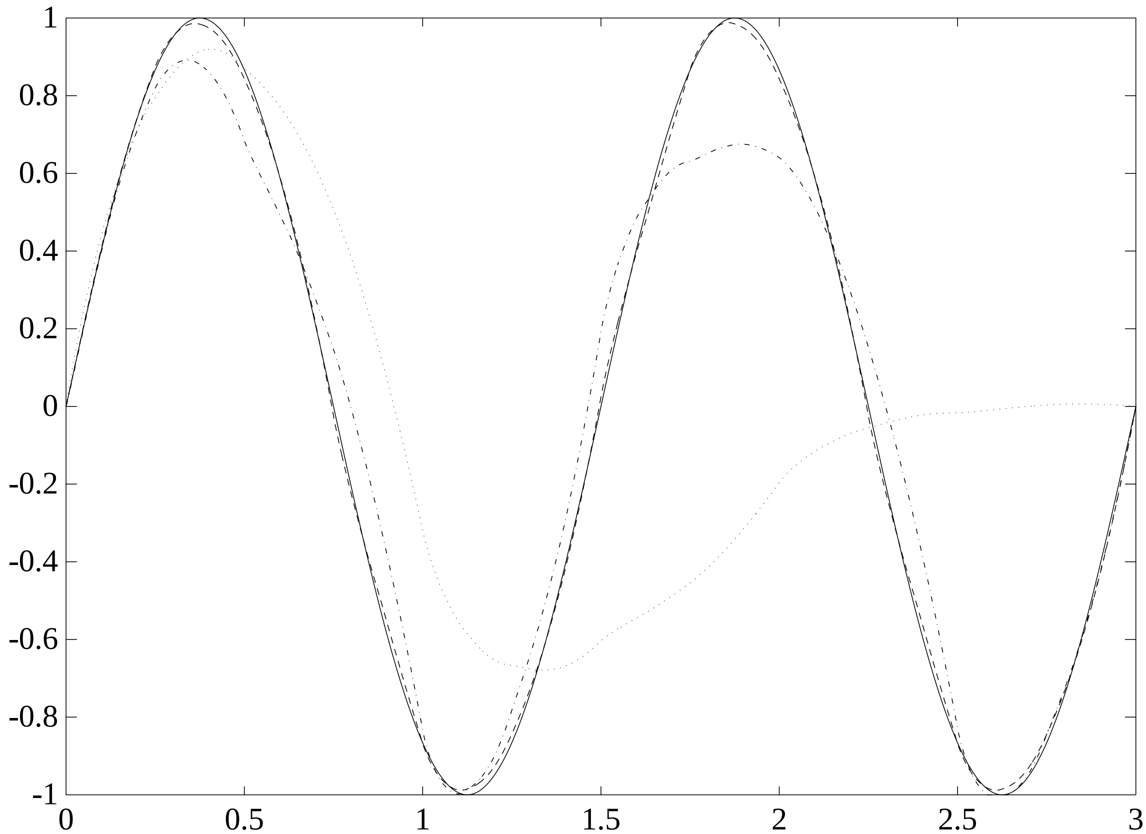

Figure 6. $\cdots\ u_0$; $\text{-}\cdot\text{-}\ u_1$; $\text{- - -}\ u_2$; — u.

for this issue, so we simply consider the problem $-u'' = \xi^2 \sin(\xi x)$. Galerkin projections u_J with $0 \le J \le 2$ are plotted in Figure 6. Visually there is no difference between u and u_3 at this scale.

Finally, let us try a Neumann boundary value problem:

$$\begin{cases} -u'' = x - \frac{3}{2}, & \text{for } x \in (0,3); \\ u'(0) = u'(3) = 0. \end{cases} \tag{5.4}$$

Now let S_J be constructed by the basis functions Ψ_{jk} defined in (4.10). The convergence rates are also as predicted by Theorem 4.1, as seen in Table 3.

(5.4)	$h^{-3}\|u-u_J\|_{0,\Omega}$	$h^{-2}\lvert u-u_J\rvert_{1,\Omega}$	CG
$J=0$	0.03052	0.1582	3
$J=1$	0.02989	0.1768	5
$J=2$	0.02961	0.1857	6
$J=3$	0.02977	0.1911	7
$J=4$	0.02930	0.1916	9

Table 3.

REFERENCES

[1] O. Axelsson and G. Lindskog, *On the rate of convergence of the preconditioned conjugate gradient method,* Numer. Math., 48 (1986), pp. 499–523.

[2] A. P. Calderón, *Intermediate spaces and interpolation, the complex method,* Studia Math., 24 (1964), pp. 113–190.

[3] P. G. Ciarlet, "The finite element methods for elliptic problems," North-Holland, Amsterdam, 1978.

[4] R. Coifman and G. Weiss, "Analyse Harmonique non commutative sur certains espaces homogènes," Springer-Verlag, Berlin, 1971.

[5] J. M. Combes, A. Grossmann and Ph. Tchamitchian, eds., "Wavelets, time-frequency methods and phase space, 2nd ed.," Springer-Verlag, Berlin, 1990.

[6] E. Cortina and S. M. Gomes, *A wavelet based numerical method applied to free boundary problems,* IPE Technical Report, Sao Jose dos Campos, Brasil, 1989.

[7] I. Daubechies, *Orthonormal bases of compactly supported wavelets,* Comm. Pure Appl. Math., 41 (1988), pp. 909–996.

[8] I. Daubechies and J. Lagarias, *Two-scale difference equations.* I. *Global regularity of solutions, and –.* II. *Infinite matrix products, local regularity and fractals,* AT&T Bell Laboratories, preprint.

[9] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series,* Trans. Am. Math. Soc., 72 (1952), pp. 341–366.

[10] R. Glowinski, W. M. Lawton, M. Ravachol and E. Tenenbaum, *Wavelets solution of linear and nonlinear elliptic, parabolic and hyperbolic problems in one space dimension,* preprint, AWARE Inc., Massachusetts, 1989.

[11] G. H. Golub and C. F. Van Loan, "Matrix computations," 2nd ed., Johns Hopkins University Press, Baltimore, 1988.

[12] P. Goupillaud, A. Grossmann and J. Morlet, *Cycle-octave and related transforms in seismic signal analysis,* Geoexploration, 23 (1984/85), pp. 85–102.

[13] A. Grossmann, M. Holschneider, R. Kronland-Martinet and J. Morlet, *Detection of abrupt changes in sound signals with the help of wavelet transforms,* in "Inverse problems: an interdisciplinary study," Academic Press, London-Orlando, 1987, pp. 289–306.

[14] A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape,* SIAM J. Math., 15 (1984), pp. 723–736.

[15] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme,* Math. Ann., 69 (1910), pp. 331–371.

[16] C. Heil, *Wavelets and frames,* in "Signal processing, pt. 1, 2nd ed.," Spring-Verlag, New Yord, 1990, pp. 147–160.

[17] R. Kronland-Martinet, J. Morlet and A. Grossmann, *Analysis of sound patterns through wavelet transforms,* to appear.

[18] A. Latto and E. Tenenbaum, *Les ondelettes á support compact et la solution numérique de l'équation de Burgers,* preprint, AWARE Inc., Massachusetts, 1990.

[19] J. S. Liénard, "Speech analysis and reconstruction using short-time, elementary waveforms," LIMSI-CNRS, Orsay, France.

[20] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$,* Trans. Amer. Math. Soc., 315 (1989), pp. 69–87.

[21] S. Mallat, *Multifrequency channel decompositions of images and wavelet models,* IEEE Trans. Acoust. Speech Signal Process, 37 (1989), pp. 2091–2110.

[22] Y. Meyer, "Principe d'incertitude, bases hilbertiennes et algebres d'operateurs," Bourbaki Seminar, no. 662, 1985.

[23] Y. Meyer, "Ondelettes et opérateurs, I. Ondelettes," Hermann, Paris, 1990.

[24] X. Rodet, *Time-domain formant-wave-function synthesis,* Computer Music J., 8 (1985),

[25] G. Strang, *Wavelets and dilation equations: A brief introduction,* SIAM Review, 31 (1989), pp. 614–627.

[26] J. Stromberg, *A modified Franklin system and higher-order systems of* $\mathbb{R}^n$ *as unconditional bases for Hardy spaces,* in "Conference on harmonic analysis in honor of Antoni Zygmund," Wadsworth, Belmont, 1983, pp. 475–493.

[27] H. Yserentant, *On the multi-level splitting of finite element spaces,* Numer. Math., 49 (1986), pp. 379–412.

[28] O. C. Zienciewicz, D. W. Kelly, J. Gago and I. Babuška, *Hierarchical finite element approaches, error estimates and adaptive refinement,* in "The mathematics of finite elements and applications IV," Academic Press, London, 1982, pp. 313–346.