QUADRATURES INVOLVING POLYNOMIALS AND DAUBECHIES' WAVELETS *

WEI-CHANG SHANN[1] AND JANN-CHANG YAN[2]

Abstract. Scaling equations are used to derive formulae of quadratures involving polynomials and scaling/wavelet functions with compact supports; in particular, those discovered by Daubechies. It turns out that with a few parameters, which are theoretically exact, these quadratures can be evaluated with algebraic formulae instead of numerical approximations. Those parameters can be obtained with high precision by solving well-conditioned linear systems of equations which involve matrices already seen in the literature of wavelets for other purposes.

Key words. scaling equation, wavelet, quadrature, polynomial, Neumann series

Subject classifications: AMS(MOS): 65A05, 65D30, 65F35, 42C05

1. Introduction. Because of their similarities with Fourier transforms and finite elements, wavelets are studied for the applicability of being a tool in scientific computation and numerical analysis, such as the numerical solution of differential equations. Among the recent articles on the later subject, we refer to the works by Bacry, Mallat and Papanicolaou [1], Beylkin, Coifman and Rokhlin [4], Glowinski, Lawton, Ravachol and Tenenbaum [8], Jaffard [9], Maday, Perrier and Ravel [14], and Xu and Shann [18].

One of the fundamental computing steps of these applications is the numerical quadratures of the forms $\int f(x)\phi(x)\,dx$, $\int f(x)\phi(x)\phi(x-k)\,dx$, or $\int_a^b f(x)\phi(x)\,dx$. As pointed out by Dahmen and Micchelli [5], since the derivatives of ϕ also satisfy (other) scaling equations, the techniques derived for the foregoing integrals can be carried to those with derivatives of ϕ. Beylkin, Coifman and Rokhlin [2] studied numerical methods for the first two kinds of integrals. There the idea is that $\phi(x)$ can be related to $\phi(2^j x-k)$ by the fast pyramid algorithm [13]; and the integral $\int f(x)\phi(2^j x-k)\,dx$ can be approximated by numerical quadrature rules if j is large enough (that is, if the support of $\phi(2^j x)$ is narrow enough). The fundation for the derivation of such quadrature points and parameters is the integrals for which $f(x)$ are polynomials. That is, the moments

$$\int x^m\phi(x)\,dx, \tag{1.1}$$

and the integrals

$$\int x^m\phi(x)\phi(x-k)\,dx, \tag{1.2}$$

$$\int_a^b x^m\phi(x)\,dx. \tag{1.3}$$

In [2], (1.1) were approximated by a recursively generated sequence of vectors, based on the 2π-periodic characteristic function of $\hat{\phi}(\xi)$. Sweldens and Piessens [17] went on to show a better

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[1] Department of Mathematics, National Central University, Chung-Li, Taiwan, R.O.C. This work was supported by the National Science Coucil in Taiwan; grant # 82-0208-M-008-058. E-mail address: shann@math.ncu.edu.tw.

[2] E-mail address: yan@math.ncu.edu.tw

approximation result for the one-point rule in [2], and to extend the multi-point rules. They also started with the relation , which states that (1.1) can be done theoretically exact. Beylkin [3] found the integrals (1.2) are associated with matrices. In particular, he found the values of $\int \phi'(x)\phi'(x-k)\,dx$ consist a normalized eigenvector of a certain matrix. This technique turns out to be a special case of the theorems in [5], which guarateed this sort of integrals always associate with eigenproblems and there are always unique solutions with appropriate assumptions.

In this article we extend the idea to treat (1.3). We also show that the matrices associated with these integrals have uniform bounds in vector ℓ^2-norm. That is, the upper bound of the norms do not change with the order of wavelets. In particular, the assocated linear systems can be solved with high precision efficiently by Neumann series.

Wavelets in this article are those discovered by Daubechies [6], which have compact supports and form an orthonormal basis of $L^2(\mathbb{R})$. In this case, the *wavelet* $\psi(x)$ is derived from the *scaling function* $\phi(x)$ which is normalized to have $\int \phi = 1$ and satisfies the *scaling equation*

$$\phi(x) = \sum_k c_k \phi(2x-k). \tag{1.4}$$

The coefficients c_k were computed by constructing a certain trigonometric polynomial $m_0(\xi) = \frac{1}{2}\sum c_k e^{-ik\xi}$, see Daubechies' "Ten Lectures" [7]. However, Strang pointed out [16] that they can also be solved by the following relations:

$$c_k = 0 \text{ for } k \notin \{0, 1, \ldots, 2p-1\}, \tag{1.5}$$

$$\sum_k c_k - 2 = 0, \tag{1.6}$$

$$\sum_k (-1)^k k^m c_k = 0, \quad \text{for } 0 \le m \le p-1, \quad (0^0 := 1) \tag{1.7}$$

and

$$\sum_k c_k c_{k-2m} - 2\delta_{0m} = 0. \tag{1.8}$$

Here p is a given positive integer. We shall call respectively such defined c_k, ϕ and ψ the *scaling coefficients*, scaling functions, and wavelets of order p. This parameter p plays several roles in the theory of wavelets. What we need now is that $\mathrm{supp}\phi = [0, 2p-1]$.

Once $\phi(x)$ is determined, the multiresolution approach asserts that

$$\psi(x) = \sum_k (-1)^k c_{1-k} \phi(2x-k) \tag{1.9}$$

will be a wavelet. One of the properties of ψ is that $\int x^m \psi\, dx = 0$ for $0 \le m \le p-1$.

Reducing the computations involving wavelets to the manipulation of c_k is a general strategy. It is sought for every kind of application for wavelets. Examples have been seen in numerical algorithms and signal decomposition [2, 12], among others. Thus the numerical values of c_k heavily determine the accuracy of these algorithms.

In the next section we discuss the computation of scaling coefficients. Then we derive the formulae for $\int x^m \phi$. As the immediate applications, polynomials $\sum k^m \phi(x-k)$, for $0 \le m \le p-1$, are constructed and formulae for $\int x^m \psi$ are derived. Next we evaluate the integrals on the half-lines $[0,\infty)$ or $(-\infty, 0]$. Finally we present that the similar idea can treat convolutions $\int x^m \phi(x)\phi(x-k)$.

2. Computation of scaling coefficients. Note that (1.8) contributes exactly $p-1$ equations in the system, namely $1 \le m \le p-1$: The situations for $m \ge p$ and $m \le -p$ are consequences from (1.5); those for $1-p \le m \le -1$ are dual to the cases of $|m|$; and the case of $m=0$ can be derived from (1.6), the $m=0$ case of (1.7), and the $m \ne 0$ part of (1.8).

Therefore, we can write conditions (1.6)–(1.8) into $2p$ equations in $\boldsymbol{x} = (x_0, \ldots, x_{2p-1})^T$. Let it be a vector-valued function $F_p(\boldsymbol{x})$. Algebraic solutions for $p = 1, 2$ and 3 are known; but it is not likely to find those with higher p. Numerical solutions for F_p, $2 \le p \le 10$, are listed in a table [7, p. 195] with 16 digits.[3]

Let $\boldsymbol{d}_{10}$ be the solution of F_{10} listed in the table (with (1.6) changed to $\sqrt{2}$). We have $\|F_{10}(\boldsymbol{d}_{10})\|_2 \approx 8 \times 10^{-6}$. This is due to the difficulty introduced in (1.7). For $p=10$ and $m=9$, the smallest and the largest number in that equation is different by a scale of 10^8.

Observe that the set of equations in (1.7) can be replaced by

$$\sum_k (-1)^k (k-\ell)^m c_k = 0, \quad \text{for } 0 \le m \le p-1. \tag{2.1}$$

for any fixed integer ℓ. In particular, we take $\ell = p$ to have the most advantage. With (1.7) replaced by (2.1), the foregoing difficulty is slightly improved. Let $\tilde{F}_p(\boldsymbol{x})$ be the alternate system of equations. One can write down the Jacobian of $\tilde{F}_p$ and perform the Newton's iteration to solve $\tilde{F}_p(\boldsymbol{x}) = \mathbf{0}$ (we use the roots of $\tilde{F}_{p-1}$ appended with two zeros as the initial guess). Our FORTRAN code is able to squeeze out more information about the scaling coefficients. First, let $\boldsymbol{c}_{10}$ be the numerical solution of $\tilde{F}_{10}$, we have $\|F_{10}(\boldsymbol{c}_{10})\|_2 \approx 2 \times 10^{-8}$. Secondly, we can calculate the scaling coefficients up to $p = 14$. Some results are listed in Table 1. All values quoted hereafter are computed in quartic precision on a VAX 6510.

Numerical values of c_k of order $2 \le p \le 14$, together with those values and programs mentioned later in this article, are available from the first author by request. It makes more sense to query and obtain the data electronically.

3. Integrals on half lines. While solving differential equations, the domain of interest is usually a bounded interval, on which boundary conditions are imposed. In this situation, integrals like $\int_a^b q(x)\phi(2^j x - k)\,dx$ will come in order. In practice one can rescale the problem and only use those wavelets and scaling functions in such fine scales that in the support of each $\phi(2^j x - k)$ (or $\psi(2^j x - k)$), there lies at most one boundary point. Since the situations are analogous at either boundary, we now suppose $x = 0$ is the left boundary point and consider the integrals on $[0, \infty)$.

As before, the core formulae for this kind of integrals are

$$N^p_{m,k} = \int_0^\infty x^m \phi(x-k)\,dx.$$

For the translation parameters $k \le 1-2p$, the integrals are zero. For $k \ge 0$, integration over $[0,\infty)$ is the same as over $\mathbb{R}$. Thus $N^p_{m,k} = M^p_{m,k}$. We now concentrate on the cases that $2-2p \le k \le -1$.

Again, we start from (1.4) with a changing of variable:

$$\int_0^\infty x^m \phi(x-k)\,dx = \frac{1}{2^{m+1}} \sum_\ell c_\ell \int_0^\infty x^m \phi(x-(2k+\ell))\,dx. \tag{3.1}$$

[3] Over there, equation (1.6) is replaced by $\sum c_k = \sqrt{2}$. Here we follow the notation in [16].

If $1-p \le k \le -1$; that is, over half of the support is still inside of $[0,\infty)$. Then the supports of $\phi(x-(2k+\ell))$, for $-2k \le \ell \le 2p-1$, actually lies entirely in $[0,\infty)$. That means, for these ℓ, the integrals on the right hand side of (3.1) can be evaluated as in Section 3. Over all, we have

$$N^p_{m,k} = \frac{1}{2^{m+1}}\Big[\sum_{\ell<-2k} c_\ell N^p_{m,2k+\ell} + \sum_{\ell\ge -2k} c_\ell M^p_{m,2k+\ell}\Big].$$

Suppose the part of $\ell \ge -2k$ is known, we have a linear system in $N^p_{m,k}$.

Given a $p \ge 2$ (there is nothing to do with $p=1$) and fix an $m \ge 0$, we have unknowns $N^p_{m,k}$ for $2-2p \le k \le -1$. Now let

$$\begin{cases} x_k = N^p_{m,-k}, & \\ b_k = \dfrac{1}{2^{m+1}} \displaystyle\sum_{\ell=2k}^{2p-1} c_\ell M^p_{m,\ell-2k}, & \end{cases} \qquad k = 1,2,\dots,2p-2.$$

Let $\boldsymbol{b} = (b_k)^T$ and $A = (a_{k\ell})$ be the matrix of order $2p-2$ such that $a_{k\ell} = c_{2k-\ell}$. Then $\boldsymbol{x} = (x_k)^T$ is the solution of the system

$$(I - \frac{1}{2^{m+1}}A)\,\boldsymbol{x} = \boldsymbol{b}.$$

Note that A is independent of m and it is a finite submatrix of the low pass filter matrix L defined in [16], in which it is also shown that A has an eigenvalue $\lambda = 1$ with an associated eigenvector $(\phi(1),\phi(2),\dots,\phi(2p-2))^T$. Here we claim another property for A.

Proposition 3.1. *Let A be the matrix defined above. $\|\cdot\|_2$ be the matrix norm induced by the Euclidean norm. We have*

$$\|A\|_2 \le \sqrt{2}.$$

Proof. Since $A = \begin{pmatrix} S \\ T \end{pmatrix}$ where

$$S = \begin{pmatrix} c_1 & c_0 & & 0 \\ & & \ddots & \\ c_{2p-3} & c_{2p-4} & \cdots & c_0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} c_{2p-1} & c_{2p-2} & \cdots & c_2 \\ & \ddots & & \\ 0 & & c_{2p-1} & c_{2p-2} \end{pmatrix}.$$

By (1.8), $ST^T = TS^T = 0$. Thus

$$AA^T = \begin{pmatrix} SS^T & 0 \\ 0 & TT^T \end{pmatrix}.$$

Also by (1.8), we have $SS^T + TT^T = 2I$. Thus if λ is an eigenvalue of SS^T with an associated eigenvector $\boldsymbol{v}$, then λ and $2-\lambda$ are eigenvalues of AA^T with associated eigenvectors $\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{v} \end{pmatrix}$, respectively. Since SS^T and TT^T are positive semi-definite, we have $0 \le \lambda \le 2$. Therefore $\|AA^T\|_2 \le 2$. ∎

Thus, for any $m \geq 0$ and $p \geq 2$, the *Neumann series*

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^{m+1}}A\right)^n \tag{3.2}$$

is an efficient method for computing the inverse of $I - \frac{1}{2^{m+1}}A$. We have prepared values for $2 \leq p \leq 14$ and $0 \leq m \leq 14$. Those lower order values are listed in Table 2. (The right most integer in each column is the exponent.) The iterations are computed in quartic precision with the stop criterion being that the ℓ^2-norm of error $< 10^{-24}$.

As for the right boundary point, one can translate the problem to the integral on $(-\infty, 0]$. Clearly

$$\int_{-\infty}^{0} x^m \phi(x-k)\,dx = M^p_{m,k} - N^p_{m,k}.$$

4. Convolutions. Now we consider the integrals of the form $\int q(x)\phi(2^j x - \ell)\phi(2^k x - m)\,dx$. If $j < k$, apply the scaling equation (1.4) recursively on $\phi(2^j x - \ell)$ will lift the scale to the same level k. Then a change of variable can go back to level $j = k = 0$. Thus in practice we do not need the formula for those ϕ's on different scales.

Again, after changing of variables, we can concentrate on the core:

$$L^p_{m,k} = q(x)\phi(x) * \phi(-x)\ (k) = \int x^m \phi(x)\phi(x-k)\,dx$$

Since $L^p_{0,k} = \delta_{0k}$, we start from $m = 1$. The support of $\phi(x)$ of order p is $[0, 2p-1]$, hence for each $p \geq 1$ and $m \geq 1$, we have unknowns $L^p_{m,k}$ for $2 - 2p \leq k \leq 2p - 2$. Then

$$\begin{aligned} L^p_{m,k} &= \int x^m \sum_n c_n \phi(2x-n) \sum_\ell c_\ell \phi(2x - 2k - \ell)\,dx \\ &= \tfrac{1}{2}\textstyle\sum_{n,\ell} c_n c_\ell \int (\frac{x+n}{2})^m \phi(x)\phi(x+n-2k-\ell)\,dx \\ &= \frac{1}{2^{m+1}}\Big[\sum_{n,\ell} c_n c_\ell L^p_{m,2k+\ell-n} + \sum_{n,\ell} c_n c_\ell \sum_{r=1}^{m}\binom{m}{r} n^r L_{m-r,2k+\ell-n}\Big]. \end{aligned} \tag{4.1}$$

The problem is recursively equivalent to a system of linear equations

$$(I - \frac{1}{2^{m+1}}A)\boldsymbol{x} = \boldsymbol{b}. \tag{4.2}$$

Let $N = 2p - 2$, then $\boldsymbol{x} = (L^p_{m,-N}, \ldots, L^p_{m,N})^T$, $\boldsymbol{b} = (b_k)$ and $A = (a_{kj})$ with

$$b_k = \frac{1}{2^{m+1}} \sum_{n=1}^{2p-1} c_n \sum_{\ell=\ell_1}^{\ell_2} c_\ell \sum_{r=1}^{m}\binom{m}{r} n^r L^p_{m-r,2k+\ell-n}, \quad \text{where } \begin{cases} \ell_1 = \max\{n-2k-N, 0\}, \\ \ell_2 = \min\{n-2k+N, 2p-1\} \end{cases}$$

and

$$a_{kj} = \sum_n c_n c_{n+j-2k}, \qquad -N \leq j, k \leq N.$$

This matrix A was considered by Lawton [11] for a sufficient and necessary condition on c_k (and hence $m_0(\xi)$) for constructing orthonormal wavelets basis. It is noted that A has an eigenvalue 2 with an associated eigenvector $\boldsymbol{v} = (\delta_{0k})$. The condition is that 2 is a nondefective eigenvalue of A. Daubechies [7] summerized it as one of the three equivalent conditions.

Since $\phi(x)$ is infinitely many times differentiable as $p \to \infty$, differentiate (1.4) to have

$$\phi^{(s)}(x) = 2^s \sum_k c_k \phi^{(s)}(2x-k).$$

Apply this equation into $\int \phi(x)\phi^{(s)}(x-k)\,dx$ and repeat the steps in (4.1), we find that the matrix $A = A(p)$ will asymptotically have eigenvalues 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, ..., as $p \to \infty$.

Actually A has a simple pattern. On the even-numbered columns, by (1.8), all entries are 0 except $a_{k,2k} = 2$. On the odd-numbered columns all entries are 0 except for $\lceil \frac{j-2p}{2} \rceil \le k \le \lfloor \frac{j+2p}{2} \rfloor$, which is the $2p$-element vector

$$\boldsymbol{h} = \begin{pmatrix} \sum_k c_k c_{k+(2p-1)} \\ \vdots \\ \sum_k c_k c_{k+3} \\ \sum_k c_k c_{k+1} \\ \sum_k c_k c_{k+1} \\ \sum_k c_k c_{k+3} \\ \vdots \\ \sum_k c_k c_{k+(2p-1)} \end{pmatrix} = \begin{pmatrix} \int \phi(x)\phi(x-\frac{2p-1}{2})\,dx \\ \vdots \\ \int \phi(x)\phi(x-\frac{3}{2})\,dx \\ \int \phi(x)\phi(x-\frac{1}{2})\,dx \\ \int \phi(x)\phi(x-\frac{1}{2})\,dx \\ \int \phi(x)\phi(x-\frac{3}{2})\,dx \\ \vdots \\ \int \phi(x)\phi(x-\frac{2p-1}{2})\,dx \end{pmatrix}.$$

Proposition 4.1. *Let A be the matrix defined in this section. $\|\cdot\|_2$ be the matrix norm induced by the Euclidean norm. We have*

$$\|A\|_2 \le 2\sqrt{2}.$$

Proof. Define matrices

$$B = \begin{pmatrix} c_0 & c_1 & & \cdots & c_{2p-1} & & & \\ & c_0 & c_1 & \cdots & c_{2p-2} & c_{2p-1} & & \\ & & \ddots & & & & & \\ & & & c_0 & c_1 & & \cdots & c_{2p-1} \end{pmatrix}_{(2N+1)\times 2(N+P)}$$

and

$$D = \begin{pmatrix} c_{2p-2} & c_{2p-1} & & & & \\ & \ddots & & & & \\ & & c_0 & c_1 & c_2 & c_3 \\ & & & & c_0 & c_1 \end{pmatrix}_{(2N+1)\times 2(N+P)}.$$

That is, $B_{ij} = c_{j-i}$ and $D_{ij} = c_{(2p-1)+j-2i}$. Observe that

$$A = D\,B^T.$$

By (1.8), DD^T is a diagonal matrix and clearly its spectrum radius is 2. Hence $\|D\|_2 = \sqrt{2}$.

Let $\boldsymbol{x}$ be a $2(N+P)$ dimensional vector. Extend $\boldsymbol{x}$ to $(\tilde{x}_k) \in \ell^2$ such that

$$\tilde{x}_k = \begin{cases} x_k & \text{if } 1 \le k \le 2(N+P) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \|B\boldsymbol{x}\|_2^2 &= \sum_{i=1}^{2N+1} \Big(\sum_j c_{j-i} \tilde{x}_j \Big)^2 \\ &= \sum_{i=1}^{2N+1} \sum_{j,k} c_{j-i} c_{k-i} \tilde{x}_j \tilde{x}_k \\ &= \sum_{i=1}^{2N+1} \sum_{j,k} c_j c_k \tilde{x}_{j+i} \tilde{x}_{k+i} \\ &= (\sum_{j,k} c_j c_k)\, (\sum_{i=1}^{2N+1} \tilde{x}_{j+i} \tilde{x}_{k+i}) \\ &\le (\sum_k c_k)^2\, (\sum_{i=1}^{2N+1} \tilde{x}_{j+i}^2)^{1/2}\, (\sum_{i=1}^{2N+1} \tilde{x}_{k+i}^2)^{1/2} \\ &\le 4\, \|\boldsymbol{x}\|_2^2. \end{aligned}$$

Therefore $\|B\|_2 \le 2$. ∎

Fortunately the linear system (4.2) is not needed for $m = 0$ (actually $\boldsymbol{b} = \boldsymbol{0}$ and $I - \frac{1}{2}A$ is singular). Hence for all cases with $p \ge 2$ ($p = 1$ is an empty case) and $m \le 1$, equation (4.2) can be solved by the Neumann series (3.2).

As in the previous section, we list values of $L^p_{m,k}$ with smaller p and m on Table 3. For each m, values for $k = 2-2p, \dots, 2p-2$ are listed from top down and from left to right. Except those for $m = 1$, only those for $k = 2-2p, \dots, 0$ are listed. Because

$$L^p_{1,k} = \int (x+k)\phi(x+k)\,\phi(x)\,dx = \int x\phi(x)\phi(x+k)\,dx = L^p_{1,-k}.$$

The integrals

$$\int_0^\infty x^m \phi(x)\phi(x-k)\,dx \text{ and } \int_{-\infty}^0 x^m \phi(x)\phi(x-k)\,dx$$

can be evaluated by the combination of techniques described in Sections 4 and 5. It is slightly tedious but otherwise straight forward.

REFERENCES

1 E. Bacry, S. Mallat and G. Papanicolaou, A wavelet based space-time adaptive numerical method for partial differential equations, *RAIRO Math Modelling Numer. Anal.,* 26: 793–834 (1992)

2 G. Beylkin R. Coifman and V. Rokhlin, Fast wavelet transforms and numerical algorithms, *Comm. Pure Appl. Math.,* 44: 141–183 (1991)

3 G. Beylkin, On the presentation of operators in bases of compactly supported wavelets, *SIAM J. Numer. Anal.,* 6: 1716–1740 (1992)

4 G. Beylkin, R. Coifman and V. Rokhlin, Wavelets in numerical analysis, in *Wavelets and their applications,* Jones and Barlett Publishers, Boston, 1992.

5 W. Dahmen and C. A. Micchelli, Using the refinement equation for evaluating integrals of wavelets, *SIAM J. Num. Anal.,* 30: 507–537 (1993)

6 I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.,* 41: 909–996 (1988)

7 I. Daubechies, *Ten lectures on wavelets,* SIAM, Philadelphia, 1992.

8 R. Glowinski, W. M. Lawton, M. Ravachol and E. Tenenbaum, Wavelets solution of linear and nonlinear elliptic, parabolic and hyperbolic problems in one space dimension, in *Computing methods in applied sciences and engineering,* SIAM, Philadelphia, 1990.

9 S. Jaffard, Wavelet methods for fast resolution of elliptic problems, *SIAM J. Numer. Anal.,* 29: 965–986 (1992)

10 B. Jawerth and W. Sweldens, An overview of wavelet based multiresolution analysis, *Math. Rev.* to appear.

11 W. Lawton, Necessary and sufficient conditions for constructing orthonormal wavelet bases, *J. Math. Phys.,* 32: 57–61 (1991)

12 S. Mallat, Multifrequency channel decompositions of images and wavelet models, *IEEE Trans. Acoust. Speech Signal Process,* 37: 2091–2110 (1989)

13 S. Mallat, A theory for multiresolution signal decomposition: The wavelet representation, *IEEE Trans. on Patt. Anal. Mach. Intell.,* 11: 674–693 (1989)

14 Y. Maday, V. Perrier and J.-C. Ravel, Adaptativité dynamique sur bases d'ondelettes pour l'approximation d'équations aux dérivées partielles, *C. R. Acad. Sci. Paris, Série I,* 312: 405–410 (1991)

15 V. Perrier, Towards a method for solving partial differential equations using wavelet bases, in *Wavelets, Time-Frequency Methods and Phase Space,* Springer-Verlag, Berlin, 1989.

16 G. Strang, Wavelets and dilation equations: A brief introduction, *SIAM Review,* 31: 614–627 (1989)

17 W. Sweldens and R. Piessens, Quadrature formulae for the calculation of the wavelet decomposition, *SIAM J. Numer. Anal.* to appear.

18 J. Xu and W.-C. Shann, Galerkin-wavelets methods for two-point boundary value problems, *Numer. Math.,* 63: 123–144 (1992)

p	c_k mantissa	E
9	.5385034958932560	−1
	.3448343038139558	0
	.8553490643594151	0
	.9295457143662946	0
	.1883695495063675	0
	−.4147517618018769	0
	−.1369535490247661	0
	.2100683422790123	0
	.4345267546122908	−1
	−.9564726412019405	−1
	.3548928132330378	−3
	.3162416585251172	−1
	−.6679620226277154	−2
	−.6054960575090131	−2
	.2612967280494491	−2
	.3258146713521844	−3
	−.3563297590215553	−3
	.5564551403430971	−4
14	.9137450851830225	−2
	.8819708777894742	−1
	.3604127050867402	0
	.7839065225916112	0
	.8926344166091910	0
	.3092470523220979	0
	−.3842256353741171	0
	−.3083459751685719	0
	.1957203884151344	0
	.1979743658370164	0
	−.1226807801540286	0
	−.1011855032474511	0
	.7811709310229835	−1
	.3815747356097654	−1
	−.4268853353342003	−1
	−.7940879199227489	−2
	.1808707483312808	−1
	−.1055313015123693	−2
	−.5444211497395586	−2
	.1501457932315697	−2
	.1001293118766936	−2
	−.5470629862491747	−3
	−.5908194756705494	−4
	.9723431362318709	−4
	−.1461902142590771	−4
	−.6207980206914902	−5
	.2439510864926535	−5
	−.2527397581044947	−6

p	c_k mantissa	E
10	.3771715759224137	−1
	.2661221827938417	0
	.7455750714864667	0
	.9736281107336398	0
	.3976377417690173	0
	−.3533362017941126	0
	−.2771098787209663	0
	.1801274485333933	0
	.1316029871010700	0
	−.1009665711967794	0
	−.4165924808760161	−1
	.4696981409739712	−1
	.5100436967814477	−2
	−.1517900233585649	−1
	.1973325364963205	−2
	.2817686590194676	−2
	−.9699478398564109	−3
	−.1647090060907779	−3
	.1323543668511067	−3
	−.1875841562750040	−4
13	.1301378205356890	−1
	.1171834948801636	0
	.4412294302098169	0
	.8641634720761928	0
	.8328156172438957	0
	.1230163936961295	0
	−.4454389578655350	0
	−.1761781021839043	0
	.2538175056505232	0
	.1031653713380694	0
	−.1496345686437708	0
	−.3746026368308236	−1
	.7939320989975462	−1
	.3365789039762014	−2
	−.3370271837915876	−1
	.5549291214847237	−2
	.1026095293477805	−1
	−.3905932326122354	−2
	−.1860643889858553	−2
	.1318508258827316	−2
	.6965217480115781	−4
	−.2335276551575616	−3
	.4338600391845477	−4
	.1476711983144362	−4
	−.6647392733914135	−5
	.7382244432298444	−6

p	c_k mantissa	E
11	.2643772943331339	−1
	.2037415352019047	0
	.6362543484607788	0
	.9697075366263585	0
	.5826055977806042	0
	−.2294918523552940	0
	−.3878209827909993	0
	.9339973813553302	−1
	.2118661798363596	0
	−.6573258290450751	−1
	−.9395863179750422	−1
	.4431450956595947	−1
	.2947348959828863	−1
	−.2172913810898397	−1
	−.4724687928191522	−2
	.6969835090237715	−2
	−.4364162061878611	−3
	−.1262925592606739	−2
	.3523548779078849	−3
	.7698847776288891	−4
	−.4898126436981186	−4
	.6355863635892411	−5
12	.1854353303644851	−1
	.1549501090023906	0
	.5336607500510726	0
	.9294193466459292	0
	.7295736544375119	0
	−.6330569419608649	−1
	−.4471438574273437	0
	−.3362894811469263	−1
	.2580637193457186	0
	.7579576121952363	−2
	−.1363756120888714	0
	.1534298714773601	−1
	.5875530909966451	−1
	−.1727977922831541	−1
	−.1815966914749901	−1
	.9491492922092193	−2
	.3180010856665811	−2
	−.3082283576704616	−2
	.9256209085601843	−5
	.5496384324992037	−3
	−.1251637115679920	−3
	−.3428272278248089	−4
	.1806933911443988	−4
	−.2162434018102160	−5

Table 1. Values of $c_0, \ldots, c_{2p-1}$ computed by solving $\tilde{F}_p$, for $p = 9, \ldots, 14$.

m	$p=2$	
0	.1503206314411140	0
	-.1634603522555265	-1
1	-.3440151734930747	-1
	.1709446898202162	-2
2	-.5286145895181897	-1
	.1259182927857764	-2
3	-.4952442124883949	-1
	.5779245553569560	-3
4	-.3974276807275230	-1
	.2295688050283120	-3
5	-.2554801209349280	-1
	.7342006169042763	-4

m	$p=3$	
0	.3992584301688844	0
	-.9671144714833962	-1
	.1451327379758485	-1
	.3409109927505814	-3
1	.1151385636190443	-1
	-.3036319565296359	-2
	.2248291989744088	-3
	.2718001606123084	-5
2	-.6170077952863928	-1
	.1291823818815563	-1
	-.5661107980966027	-3
	-.3472824436190665	-5
3	-.5070109561947334	-1
	.1625872456589445	-1
	-.2773735449615905	-3
	-.8571552835851268	-6
4	.3780523354512585	-1
	.1549099146654685	-1
	.3588546447961354	-6
	.5565622041413277	-9
5	.2533667158043628	0
	.1251034227999401	-1
	.1730856790444605	-3
	.1344757307928110	-6

m	$p=4$	
0	.6200655900672920	0
	-.1533818363287321	0
	.4557208206919333	-1
	-.6365528562791968	-2
	-.5009369295617957	-3
	.3843127851498774	-5
1	.110435827929347	0
	-.3090275555376952	-1
	.7197339527354625	-2
	-.6511346389440849	-3
	-.3492623243400755	-4
	.1323990260823287	-6
2	-.4551394218976834	-1
	.1226490529745481	-1
	-.2284305377464601	-2
	.1394681006630964	-3
	.5118302714781515	-5
	-.9644556850031373	-8
3	-.8457642026662238	-1
	.2652761794633628	-1
	-.5462532496552608	-2
	.1391186620569128	-3
	.5536204964636213	-5
	-.5200806309848612	-8
4	-.1705501888165446	-1
	.1314423489691683	-1
	-.6440471798063460	-2
	.1815698685689570	-4
	.3046903751981842	-5
	-.1429074097290984	-8
5	.1462839994507927	0
	-.4814993345531820	-1
	-.6254059695094663	-2
	-.7320272693522030	-4
	.1412297222383875	-5
	-.3309605353104780	-9

m	$p=5$	
0	.7793859884434348	0
	-.1166736535477611	0
	.4373076800008569	-1
	-.1457714708355652	-1
	.1802061160837002	-2
	.2378707250628387	-3
	.2125356814537938	-5
	.4968905745904786	-8
1	.2472418662638166	0
	-.6502647707355659	-1
	.2128039859636469	-1
	-.4848030050689274	-2
	.3924076960087620	-3
	.4708353042639666	-4
	.2528130013226264	-6
	.2968364600152115	-9
2	.1683913184217972	-1
	-.5463863640264356	-2
	.2056959354870981	-2
	-.4185293464112773	-3
	.1390086453705529	-4
	.2157982259904782	-7
	.3394005943797560	-8
	.1996931165591535	-11
3	-.1041241192508033	0
	.3303513141103077	-1
	-.9735553181468904	-2
	.1596936256692592	-2
	-.5087010631646538	-4
	-.4649993783751550	-5
	-.9822349553543982	-8
	-.2892797371007685	-11
4	-.9854743290576902	-1
	.3305510938575159	-1
	-.1301979888936725	-1
	.2582297156974469	-2
	-.2039728154484655	-4
	-.3361014881749767	-5
	-.1137935616140118	-8
	-.1676608209029098	-12

Table 2. Values of $N^p_{m,k}$, index k is assumed from top down for $k=-1,-2,\ldots,2-2p$.

m	$p=2$	
1	.2209248479041935	−2
	−.7069595132934193	−1
	.7709480019161613	0
2	.1129316506252819	−2
	−.7992931762467025	−1
	.6920786996260350	0
	−.2213212202833541	0
	.9966310422420560	−2
3	.4801180105120833	−3
	−.7880279788224953	−1
	.7063258426098530	0
	−.5306786047442860	0
	.3376699879653222	−1
4	.1762139243443035	−3
	−.7547944363240092	−1
	.8243435419404370	0
	−.1153050346226788	1
	.1018167054878505	0
m	$p=3$	
1	−.3962222539037579	−5
	−.6762193133290803	−3
	.1921288312823522	−1
	−.1210432570859053	0
	.1022422278797956	1
2	−.8722668793805085	−6
	−.2785130417988106	−3
	.2525414480873315	−1
	−.1949420477992854	0
	.1177403241056208	1
	−.4370285619710961	0
	.1021056773216740	0
	−.4335828921773292	−2
	−.3257004719168114	−4
3	.5918040227649401	−7
	.7557861023437038	−4
	.2721826041841674	−1
	−.2750201367439243	0
	.1527754162967990	1
	−.1222976051399496	1
	.4092977268096384	0
	−.2068896022584009	−1
	−.2005947040240934	−3
4	.1817127192644162	−6
	.2986597899560533	−3
	.2657424957217791	−1
	−.3942661765515609	0
	.2255522127692071	1
	−.3148172038666592	1
	.1465232068432634	1
	−.8686578698390794	−1
	−.1096937991258460	−2
m	$p=4$	
1	−.7894210894573288	−8
	.6466937564834438	−5
	.1371772167646553	−3
	−.5234339264976994	−2
	.3857211606346907	−1
	−.1639889807110571	0
	.1266408348748143	1
2	−.2481058751990008	−8
	.4113957050341542	−5
	−.2081415938567333	−5
	−.7498018218541771	−2
	.7285205719738486	−1
	−.3463363606154461	0
	.1776709637169711	1
	−.6743143220375603	0
	.2271405214512611	0
	−.3890405380840373	−1
	.1095336318178675	−2
	.6878333269868592	−4
	−.9721158948686947	−7
3	−.6212732699799800	−9
	.2132369503651301	−5
	−.1665197199226888	−3
	−.8199755930384550	−2
	.1132016368360732	0
	−.6273119123481460	0
	.2785011456332524	1
	−.2158287936327655	1
	.1013179372782011	1
	−.2170090800516393	0
	.6393009693517961	−2
	.5488620426213573	−3
	−.8978551074197153	−6
4	−.1439891487282549	−9
	.1095530472749087	−5
	−.2987806458035170	−3
	−.7084205216897087	−2
	.1681395265920187	0
	−.1144290470963721	1
	.4938148658893752	1
	−.6387512206893210	1
	.4056509708048852	1
	−.1075682900800282	1
	.3195445539708277	−1
	.3894305260514225	−2
	−.7371561450969411	−5
m	$p=5$	
1	−.1161135404402930	−11
	−.4348352563597170	−8
	−.3252907262678049	−5
	−.1633452428583043	−4
	.1367924162378023	−2
	−.1237059088588241	−1
	.5811388198527233	−1
	−.2032594207818311	0
	.1506243612626073	1
2	.1013308018208450	−12
	.7677088236136883	−9
	−.2061539099002361	−5
	.3025058914947968	−4
	.2064953020168976	−2
	−.2582855877556951	−1
	.1433102361372601	0
	−.5320790768916387	0
	.2487970775549777	1
	−.9385979184553010	0
	.3757657640783495	0
	−.1000521040908639	0
	.1300834631919316	−1
	−.1330946537088246	−3
	−.4109642625113895	−4
	−.6010922706674669	−7
	−.1847683566862605	−10

Table 3. Values of $L^p_{m,k}$, index k is assumed from top down for $k = 2\neg 2p, \ldots, 2p\neg 2$, except for $m = 1$.