Machin's Method for Approximating π

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The digit hunters of Newton's time returned to the *Gregory series*

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

which they modified in various ways to accelerate its convergence. For example, in 1706, John Machin (1680–1752), Professor of Astronomy in London, used the following stratagem to make the Gregory series rapidly.

For an angle β such that $\tan\beta = 1/5$, we have

$$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{5}{12} \quad \text{and} \quad \tan 4\beta = \frac{2\tan 2\beta}{1-\tan^2 2\beta} = \frac{120}{119}$$

This differs only by $1/119$ from 1, whose arctangent is $\pi/4$; in terms of angles, this difference is

$$\tan(4\beta - \frac{\pi}{4}) = \frac{\tan 4\beta - 1}{1+\tan 4\beta} = \frac{1}{239}$$

and hence

$$\arctan\frac{1}{239} = 4\beta - \frac{\pi}{4} = 4\arctan\frac{1}{5} - \frac{\pi}{4}$$

Substituting the Gregory series for the two arctangents, Machin obtained

$$\begin{aligned}\pi &= 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239} \\ &= 16\left(\frac{1}{5} - \frac{1}{3\times 5^3} + \frac{1}{5\times 5^5} - \cdots\right) - 4\left(\frac{1}{239} - \frac{1}{3\times 239^3} + \frac{1}{5\times 239^5} - \cdots\right)\end{aligned}$$

Machin used this method to calculate π to 100 decimal places in 1706. And the programmers of ENIAC, the first electronic computer in the world, used this method to calculate π to 2,037 decimal places in September of 1949.