Reflections on Mathematical Communication

from Taiwan Math Curriculum Guideline and PISA 2003

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In this essay we present our thoughts on mathematical communication. We will confine the meaning of mathematical communication in the mathematics classrooms, and argue that if it is going to be in the mathematics curriculum then it must have clear guideline for instruction methods, teaching materials and assessments. We discuss the role of mathematical communication in Taiwan's government Mathematics Curriculum documents, and seek for practical insights from PISA 2003 test problems and grading guides. Our APEC project will be designed for grade 6 or 12 year-olds, we will consult PISA 2003 (which is designed for students of age 15) and come up with matching ideas.

Mathematical communication for the mathematics classrooms

One who lives in a society cannot avoid communication. He is either expressing himself or tries to understand expressions of other people. Education takes the responsibility to help children and young adults to grow into the society, so it is quite natural that education must take some care about communications. And the community of math educators has also some thoughtful discussions on mathematical communication in particular, starting from 1—6 graders and extended to secondary schools. We will learn much of the expertise most renowned mathematics educators in this symposium. And here we report some thoughts from the local and practical point of view.

When people talk about mathematics, when someone write about mathematics, for instance an article of public interests on *the most beautiful identity of mathematics*, that is exp(i*Pi)+1=0, or even a (Japanese) fiction on *the expression doctor loved most*, and when people read the articles or fictions, they are all some manner of communication and they are all about mathematics. However, this general sense of mathematical communication is too broad to discuss in the context of mathematics education. It might end up with pointless if we consider too much about this broad sense of mathematical communication. On the contrary, we should consider the mathematical communication as one of the competence that can be taught and learned in a curriculum. In this sense, the mathematical communication under discussion shall be a target or subject of mathematical teaching and learning. That means, among other things, it happens between teachers and pupils in a classroom, it must have instruction methods or suggestions, it better has texts, examples, role models or teaching materials, and in some sense this learned ability shall be measurable.

Mathematical communication in Taiwan's mathematics curriculum guideline

The government document for mathematics curriculum guideline (grades 1—9) has been modified or revised quite often in the recent years. The current document was originally issued in Nov, 2003, which preserved some statements about mathematical communication proposed by the previous edition. The document categorized the mathematics curriculum into five subjects; among them was the *connection* subject which had 9 targets for mathematical communication. For instance, the first target for mathematical communication was "*Understand the comprehension of the mathematical language, such as symbols, terms, tables, graphs and informal deductions.*" Such a statement of curriculum target was not clear enough for textbook writers, neither for teachers and parents, to understand and to implement. Therefore in the appendix the document composed a list of elucidation (explanations and examples) for each target.

The elucidation for each item of mathematical learning target was completed with examples and do-s and don't-s; we can say that they are quite clearly put and fairly straight forward to follow. Unfortunately the elucidations for the mathematical communication targets were not as clearly written, and hence were not very helpful. For instance the explanation for the first target was: "*Language is necessary for the communication of mathematical contents. Of course one has to understand the language in order to communicate.*"

We think the curriculum guideline (or standard) is different from an education policy or philosophy in the sense that the former is an *operational* document. It may not be as precise as a mathematics text, but it shall be at least as precise as a legal document. Textbooks, teaching materials, classroom activities and, most concerned, assessment problems, shall be able to refer to this document for their proper conducts. In case of controversial situations, the document shall be the source to settle the different points of view.

It is in the foregoing sense that the statements, together with their elucidations, for mathematical communication were considered not proper for an official curriculum document. Therefore they are temporarily concealed in the most recent revision of the document in late 2007. More knowledge, and hence more studies, are necessary to make this subject clear enough to be considered a mathematics learning subject or target. Our APEC project is expected to offer a little bit help.

Emphasizing the written mathematical communication

There are suggestions on the merits of oral mathematical communication that takes place in classrooms, either between teach and students or among students. It seems American pupils are more verbal then ours in Taiwan (it seems Japanese pupils are quite verbal in the classrooms too). Although pupils in elementary schools are much more active and willing to share their opinions then their elder brothers and sisters, it is still doubtful that the oral communication could be a norm in Taiwan classrooms. Teachers may ask students to *think out loud*, however, it is to certain degree against the value of this society. We understand that teachers might welcome pupils to think out loud, for in that case they don't have to make the efforts to probe the students' minds. However, pupils at young age may not be able to express their thinking effectively, and teachers may fail to *listen* their minds verbally.

One of the other two concerns about oral mathematical communication is: It takes time. Everybody knows that communication, especially the deeper ones, takes time. Public schools of California have almost twice the hours for mathematics then those in Taiwan. This is one of the most serious constraints for communications in depth in the classroom. According to the lesson study videos of this APEC project, Japanese teachers almost always run out of time when there are profound communications in the classroom.

And finally we also concern about the teachers training. It also takes time to educate teachers for adopting the oral communication skills, classroom management skills and time management skills.

To conclude, we think for the recent future, it is not practical to motivate oral mathematical communications in Taiwan classrooms. It is more practical, and we think it is more urgent, to motivate written mathematical communications in the mathematics curriculum (including assessments). It is one of the good traditions that are vanishing. The ultimate form of mathematical writing is to write down the statements of definitions and theorems, and to write a proof. Since the abandon of written problems in national examinations, students' abilities of writing (and also reading) are thought to degrade very soon.

There are not much formally to write for elementary school mathematics. However, a good manner is of great worth for the future. One of the aspects to emphasize is the logical and meaningful recordings of arithmetic procedures. For instance, to write a sequence of equal signs in $17\times27=17\times(30-3)=510-51=460-1=459$, pupils often fail to present the equal sign (fail to write the sign or fail to keep the quantities equal). However, in the situation of solving an equation shall be written with meaning words to connect the equations. For instance, since $4x-3=5$ amounts to $4x=5+3=8$, by dividing 4 on both sides we have $x=8/4=2$. Some pupils may add improper equal signs at wrong places, and many pupils simply write the equations consecutively without understanding what they are deducing. Teachers' writings on the blackboards or lecture notes may help a lot as role models for students to follow. And it may not take too much effort to ask teachers adopt a better (more logical and meaningful) way of mathematical writing.

Mathematical communication in the three phases of mathematisation

PISA 2003 uses the word *mathematisation* for the multi-phase process of solving problems mathematically. The front phase involves the understanding of problem, and transformation from reality situations to mathematical models. The inner phase is purely mathematical, one solves a mathematically formulated problem by mathematical methods, including calculation and deduction. The final phase is to interpret the mathematical result back into the reality.

There are ingredients of mathematical communication in the front phase. For instance, a common problem says "*What are the three consecutive even numbers with the sum 264?*" This is a mathematical problem, one needs not the transformation. A basic level of communication ability is to link the daily language *consecutive* into its mathematical meaning. And one has to understand the mathematical terms *even* and *sum*. A more interesting communication problem is when someone presents a solution by first saying "*Let n be an even number …*", and someone might ask something like "*How can you let n be an even number? What if it is odd? Why n must be even and cannot be odd?*" What can a teacher do when some student asks this kind of question?

The communication involved in the inner phase is to write arithmetic and logic expressions properly, for instance the writings demonstrated in the previous section. Although ultimately one is expected to write a mathematical proof for the communication of his mathematical reasoning, it is neither necessary nor practical to develop this level of abilities under grade 9.

Mathematical communication in PISA 2003

PISA talks about mathematical communication in some core principles of their test design. For instance, they define *mathematical literacy* as the capacity of students to analyze, reason and *communicate* effectively as they pose, solve and interpret mathematical problems in a variety of situations involving quantitative, spatial, probabilistic or other mathematical concepts. They try to design a test that measures the mathematical literacy of 15 years-olds. PISA's idea of communication focused on the final phase of mathematisation which, they think, involves some form of translation of the mathematical result into a solution that works for the original problem context, a reality check of the completeness and applicability of the solution, a reflection on the outcomes and *communication* of the results, which may involve explanation and justification or proof.

In my opinion, it is not surprise to notice the economists, enterprisers and bankers based OECD takes a more practical approach for the PISA test. They care much about the outcome, the result and the final solution (even with compromise). I am also glad to learn that PISA provided a practical definition of their idea of mathematical communication by describing their problems and grading standards.

PISA divides students into six levels of proficiency. As communication belongs to the final phase, it is supposed to be developed later then the routine operations. There is no assumption of communication abilities for levels one and two. The communication abilities start at level 3: develop short reports on their interpretations, results and reasoning, and gradually upgrade to level 6: precisely communicate their actions and reflections regarding their findings, interpretations, arguments, and the appropriateness of these to the original situations. The descriptions will be more concrete when we inspect samples of problems and grading standards.

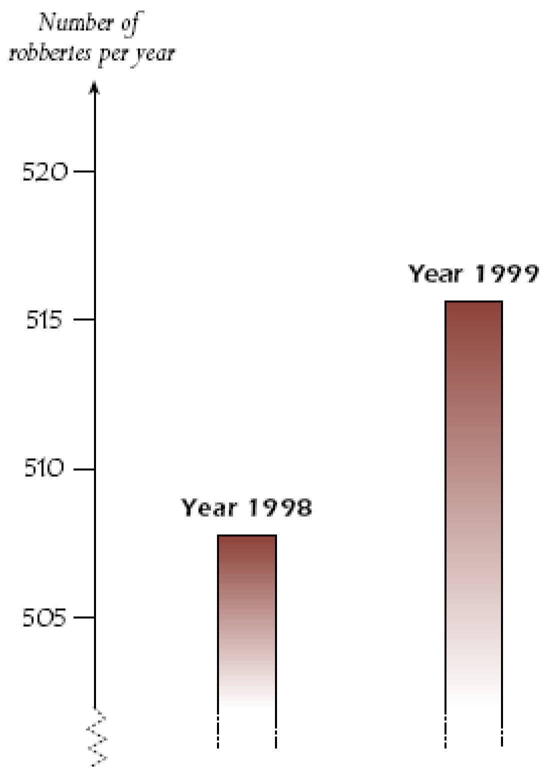
Sample 1. A problem in the subject of quantity. *Mei-Ling from Singapore was preparing to go to South Africa for 3 months as an exchange student.She needed to change some Singapore dollars (SGD) into South African rand (ZAR).* There are three problems in this set.

1. *Mei-Ling found out that the exchange rate between Singapore dollars and South African rand was:1 SGD = 4.2 ZAR. Mei-Ling changed 3000 Singapore dollars into South African rand at this exchange rate.How much money in South African rand did Mei-Ling get?* This is a level 1 problem.

2. *On returning to Singapore after 3 months, Mei-Ling had 3 900 ZAR left. She changed this back to Singapore dollars, noting that the exchange rate had changed to: 1 SGD = 4.0 ZAR. How much money in Singapore dollars did Mei-Ling get?* This similar question becomes a level 2 problem.

3. *During these 3 months the exchange rate had changed from 4.2 to 4.0 ZAR per SGD. Was it in Mei-Ling's favour that the exchange rate now was 4.0 ZAR instead of 4.2 ZAR, when she changed her South African rand back to Singapore dollars? Give an explanation to support your answer.* This is obviously a question that requires communication. It is considered level 4.

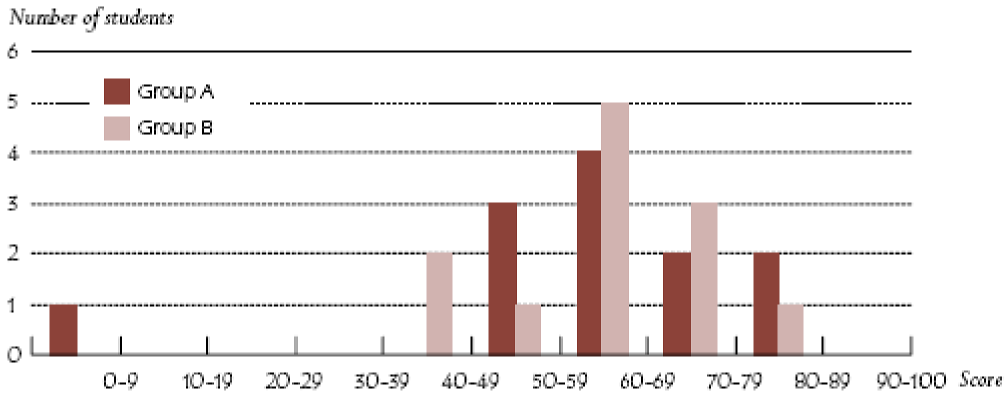
Sample 2. A problem in the subject of uncertainty. A TV reporter showed this graph and said: "The graph shows that there is a huge increase in the number of robberies from 1998 to 1999." *Do you consider the reporter's statement to be a reasonable interpretation of the graph? Give an*



explanation to support your answer.

Students are expected to say no to the reporter. If one can deny the reporter by converting the absolute counts to relative rates, or if one points out that we need more data of previous years to actually judge about the trend, then he is considered to have level 6 of proficiency. If one can say something like an increment of 8 out of 500 is not considered *huge*, he is considered level 4.

Sample 3. Another problem of uncertainty which shows PISA's idea of level 5. *The diagram shows the results on a science test for two groups, labelled as Group A and Group B. The mean score for Group A is 62.0 and the mean for Group B is 64.5. Students pass this test when their score is 50 or above. Looking at the diagram, the teacher claims that Group B did better than Group A in this test. The students in Group A don't agree with their teacher. They try to convince the teacher that Group B may not necessarily have done better. Give one mathematical argument, using the graph that the students in Group A could use.*



If one can point out the distorting effect of the outlier in the results of Group A, or he can argue that more students pass the test in Group A, then he is considered at level 5.

PISA provides a set of good examples for the assessment of mathematical communications. However, it aims at 15 years-olds. We think there are feasible materials and test problems in this manner that are suitable for students of grade 6. In our APEC project, we would like to design a lesson for teachers of 6 graders that will effectively teach mathematical communication with test problems to measure the progresses.